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KRISHNAKUMARI GANESH PRASAD PRIZE AND MEDAL

(FOR 1938)

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1

On a Class of Determinants having Geometrical Applications.

BY

N. N. Ghosh.

The determinants studied in this paper are those formed by the multiplication of rectangular arrays. If

$$A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{vmatrix}, \qquad B = \begin{vmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \dots & \dots & & \dots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{vmatrix}, \qquad \dots \quad (1)$$

be a pair of rectangular arrays each having m rows (linearly independent) and n columns, $n > m$, then the product AB or BA of these two arrays is the determinant of mth order

$$\begin{vmatrix} (a/b)_{11} & (a/b)_{12} & \dots & (a/b)_{1m} \\ (a/b)_{21} & (a/b)_{22} & \dots & (a/b)_{2m} \\ \dots & \dots & & \dots \\ \dots & \dots & & \dots \\ (a/b)_{m1} & (a/b)_{m2} & \dots & (a/b)_{mm} \end{vmatrix}. \qquad \dots \quad (2($$

In the above $(a/b)_{ij}$ represents

$$a_{i1}b_{j1} + a_{i2}b_{j2} + \dots + a_{in}b_{jn}, \qquad \dots \quad (3)$$

which may be regarded as the product of two single-rowed arrays $| a_{i1} \quad a_{i2} \dots a_{in} |$ and $| b_{j1} \quad b_{j2} \dots b_{jn} |$.

We shall use the following compact notation

$$\begin{Bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{Bmatrix}, \qquad \dots \qquad \dots \quad (4)$$

to represent (2) and the properties of the determinant will be obtained with reference to (4).

To write down the constituent in the ith row and jth column of (2) by means of (4), we take the ith element a_i of the first row and jth element b_j of the second row from it. Then we associate with a_i, the array $| a_{i1}\ a_{i2} \dots a_{in} |$ and with b_j, the array $| b_{j1}\ b_{j2} \dots b_{jn} |$ and by forming their product we get (3).

2. From each of the rectangular arrays (1) we can form $\binom{n}{m}$ determinants of order m. Let α_{mp}, β_{mp} represent a pair of *corresponding* determinants in A and B, then the value of (2) is expressed also as

$$\alpha_{m1}\beta_{m1} + \alpha_{m2}\beta_{m2} + \dots\dots + \alpha_{mp}\beta_{mp} + \dots\dots\dots \qquad \dots \quad (5)$$

where all possible products of corresponding determinants are involved.

The $n!/m!(n-m)!$ quantities $\alpha_{m1}, \alpha_{m2}, \dots \alpha_{mp} \dots\dots$ taken in some definite order will be called *co-ordinates* of the array A.

3. Square of an array A is, by our notation (4), represented as

$$\begin{Bmatrix} a_1 & a_2 & \dots, & a_m \\ a_1 & a_2 & \dots & a_m \end{Bmatrix}, \qquad \dots \qquad \dots \quad (6)$$

whence the element in the ith row and jth column of (2) is written

$$(a/a)_{ij} = a_{i1}a_{j1} + a_{i2}a_{j2} + \dots\dots + a_{in}a_{jn}\,. \qquad \dots \quad (7)$$

Corresponding to (5), the value of (6) is also given in the form

$$\alpha^2_{m1} + \alpha^2_{m2} + \dots\dots + \alpha^2_{mp} \dots\dots\dots \qquad \dots \quad (8)$$

To find an interpretation of A + B, we remark that since

$$(A+B)^2 = A^2 + B^2 + 2AB$$
$$= (\alpha_{m1} + \beta_{m1})^2 + (\alpha_{m2} + \beta_{m2})^2 + \dots + (\alpha_{mp} + \beta_{mp})^2 + \dots\dots \quad (9)$$

A + B must represent an array whose co-ordinates are the sums of the co-ordinates of A and B. As we cannot in general express the sum of two determinants as a single determinant, so with regard to arrays. A + B cannot in general be expressed in the form of an array. In the case of single-rowed arrays this is obviously possible and such arrays may be identified with vectors. From (5) and (9) it can be seen that the co-ordinates of an array form a vector.*

* Such vectors are called *pure vectors*.

4. In the determinant (4) we have only two rows and m columns, the elements in each of them are one-rowed arrays of order n, which we shall, henceforth, call vectors. There are thus two sets of vectors a_i, b_i and (4) may be called the *scalar determinant* of mth order of the sets of vectors a_i, b_i.

The following properties of a scalar determinant are easily verified:

(*i*) Interchange of rows or columns in a scalar determinant does not alter its value.

(*ii*) If any two vectors of a row be interchanged, the sign of the scalar determinant is changed. Hence, if any two vectors in a row be identical, the determinant vanishes.

(*iii*) If any vector in a scalar determinant is the sum of a certain number of vectors, the determinant can be resolved into a corresponding number of scalar determinants.

(*iv*) If any vector is multiplied by a scalar, the determinant is multiplied by that scalar.

(*v*) If $m > n$ or the vectors be not linearly independent, the scalar determinant (4) vanishes. If $m = n$, the scalar determinant breaks up into the product of two determinants.

5. We shall now obtain Laplace's expansion for a scalar determinant.

If $i_1, i_2, \dots i_r$ is a sequence of r distinct integers in natural order, chosen out of the integers $1, 2, \dots m$ and $i_{r+1} \dots\dots i_m$ the remaining $(m-r)$ integers also arranged in natural order, then the scalar determinant

$$\begin{Bmatrix} a_1 & a_2 & \dots & a_m \\ b_1 & b_2 & \dots & b_m \end{Bmatrix},$$

may be expressed in the form

$$\Sigma(-1)^{I-\frac{r(r+1)}{2}} \begin{Bmatrix} a_1 & a_2 & \dots & a_r \\ b_{i_1} & b_{i_2} & \dots & b_{i_r} \end{Bmatrix} \begin{Bmatrix} a_{r+1} & a_{r+2} & \dots & a_m \\ b_{i_{r+1}} & b_{i_{r+2}} & \dots & b_{i_m} \end{Bmatrix}, \quad \dots (10)$$

where $\qquad I = i_1 + i_2 + \dots + i_r.$

In the above $\binom{m}{r}$ terms are involved each corresponding to an r-combination of the numbers $1, 2, \dots m$.

6. The following general theorem with regard to scalar determinants may be proved by the process of induction.

If the vectors a_i $(i=1, 2, \ldots s)$ are replaced by

$$p_{i1}\, a'_1 + p_{i2}\, a'_2 + \ldots\ldots + p_{ir}\, a'_r, \quad r \not< s \qquad \ldots \quad (11)$$

in the scalar determinant

$$\begin{Bmatrix} a_1 & a_2 & \cdots & a_m \\ b_1 & b_2 & \cdots & b_m \end{Bmatrix},$$

then it is resolved into

$$\Sigma \mathrm{P}_{i_1 i_2 \ldots i_s} \begin{Bmatrix} a'_{i_1} & a'_{i_2} & \ldots & a'_{i_s} & a_{s+1} & \ldots & a_m \\ b_1 & b_2 & \ldots & b_s & b_{s+1} & \ldots & b_m \end{Bmatrix}, \qquad \ldots \quad (12)$$

where $\mathrm{P}_{i_1 i_2 \ldots i_s}$ stands for the determinant of sth order,

$$\begin{vmatrix} p_{1i_1} & p_{1i_2} & \cdots & p_{1i_s} \\ p_{2i_1} & p_{2i_2} & \cdots & p_{2i_s} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ p_{si_1} & p_{si_2} & \cdots & p_{si_s} \end{vmatrix}, \qquad \ldots \qquad \ldots \quad (13)$$

the summation being extended over all s-combinations of the numbers $1, 2, \ldots\ldots r$.

If $r=s$, then obviously

$$\begin{Bmatrix} a_1 & a_2 & \ldots & a_m \\ b_1 & b_2 & \ldots & b_m \end{Bmatrix} = \mathrm{P}_{12\ldots r} \begin{Bmatrix} a'_1 & a'_2 & \ldots & a'_r & a_{r+1} & \ldots & a_m \\ b_1 & b_2 & \ldots & b_r & b_{r+1} & \ldots & b_m \end{Bmatrix} \qquad \ldots \quad (14)$$

7. Starting from an origin O and a system of n rectangular axes, the m vectors $a_1, a_2, \ldots a_m$ forming the array A determine an m-dimensional linear sub-space. If $p_1, p_2, \ldots p_m$ be scalars, then any vector of the form $p_1 a_1 + p_2 a_2 + \ldots + p_m a_m$ belongs to the sub-space. Let us suppose that in the scalar determinant

$$\begin{Bmatrix} a_1\, a_2 \ldots a_m \\ b_1\, b_2 \ldots b_m \end{Bmatrix},$$

the vectors a_i $(i=1, 2, \ldots s)$ are replaced by vectors

$$p_{i1} a_1 + p_{i2} a_2 + \ldots\ldots + p_{im} a_m, \qquad \ldots \quad (15)$$

all belonging to the same sub-space. By the last theorem, we notice

that (12) now contains only a single term, the other terms vanishing because of the property (*ii*). Hence the result is put in the form

$$P_{12\cdots s}\begin{Bmatrix} a_1 \, a_2 \ldots a_m \\ b_1 \, b_2 \ldots b_m \end{Bmatrix}. \qquad \ldots \quad (16)$$

If again the vectors b_i $(i=1, 2, \ldots t)$ are replaced by

$$q_{i1}b_1 + q_{i2}b_2 + \ldots\ldots\ldots + q_{im}b_m,$$

in a similar manner, the resulting change will be indicated by the formula

$$P_{12\cdots s} \cdot Q_{12\cdots t}\begin{Bmatrix} a_1 \, a_2 \ldots a_m \\ b_2 \, b_2 \ldots b_m \end{Bmatrix}. \qquad \ldots \quad (17)$$

8. We shall now prove in a general way that the scalar determinant is an invariant* for any orthogonal transformation.

Let $a'_1, a'_2, \ldots a'_n$ be a set of n independent orthogonal unit vectors such that

$$(a'/a')_{ij} = 0, \quad i \neq j,$$

$$(a'/a')_{ii} = 1.$$

By choosing m of them we can form $\binom{n}{m}$ sets of vectors and it is easy to see that the scalar determinant of two similar sets is 1, while that of two dissimilar sets is 0.

$$\text{Let} \quad \begin{aligned} a_i &= p_{i1}a'_1 + p_{i2}a'_2 + \ldots\ldots\ldots + p_{in}a'_n, \\ b_i &= q_{i1}a'_1 + q_{i2}a'_2 + \ldots\ldots\ldots + q_{in}a'_n, \end{aligned} \quad (i=1, 2, \ldots m)$$

then the scalar determinant

$$\begin{Bmatrix} a_1 \, a_2 \ldots\ldots a_m \\ b_1 \, b_2 \ldots\ldots b_m \end{Bmatrix},$$

transforms into

$$\Sigma P_{i_1 i_2 \ldots i_m} Q_{i_1 i_2 \ldots i_m}\begin{Bmatrix} a'_{i_1} \, a'_{i_2} \ldots\ldots a'_{i_m} \\ a'_{i_1} \, a'_{i_2} \ldots\ldots a'_{i_m} \end{Bmatrix}, \qquad \ldots \quad (18)$$

where the sum is taken over all m-combinations of the numbers 1, 2,...n.

* It is obvious since every element of (2) is known to be invariant.

As the scalar determinants in (18) have all the same value 1, we see that this can be expressed in the form (5) and therefore as the scalar determinant

$$\begin{Bmatrix} p_1 \; p_2 \ldots\ldots p_m \\ q_1 \; q_2 \ldots\ldots q_m \end{Bmatrix},$$

which corresponds to the vectors a_i, b_i, referred to the new orthogonal basis.

9. In the original n-space, let us consider a pair of m-dimensional sub-spaces corresponding to A and B. Then the invariants

$$\begin{Bmatrix} a_1 \; a_2 \ldots\ldots a_m \\ a_1 \; a_2 \ldots\ldots a_m \end{Bmatrix} \text{ and } \begin{Bmatrix} b_1 \; b_2 \ldots\ldots b_m \\ b_1 \; b_2 \ldots\ldots b_m \end{Bmatrix},$$

measure the square of the *contents* of the m-dimensional parallelotopes formed by the vectors a_i and b_i of A and B respectively. In particular,

$$\begin{Bmatrix} a_1 \; a_2 \; a_3 \\ a_1 \; a_2 \; a_3 \end{Bmatrix},$$

is the square of the volume of the parallelopiped bounded by the vectors a_1, a_2, a_3 as conterminous edges. If further, we form an entity

$$\frac{\begin{Bmatrix} a_1 \; a_2 \ldots\ldots a_m \\ b_1 \; b_2 \ldots\ldots b_m \end{Bmatrix}}{\sqrt{\begin{Bmatrix} a_1 \; a_2 \ldots a_m \\ a_1 \; a_2 \ldots a_m \end{Bmatrix}} \sqrt{\begin{Bmatrix} b_1 \; b_2 \ldots b_m \\ b_1 \; b_2 \ldots b_m \end{Bmatrix}}},$$

then by (15) and (17), it is an invariant for any transformation of the type (15). Such invariants are called *angular invariants*.*

10. We shall now express the projection of a given vector on a given sub-space.

Let b_1 be the given vector and a_i ($i=1, 2, \ldots m$), be the vectors forming the sub-space. Then the projection of b_1 must be of the form $p_{11}a_1 + p_{12}a_2 + \ldots\ldots + p_{1m}a_m$. If the projection be orthogonal, we must have the vector

$$-b_1 + p_{11}a_1 + p_{12}a_2 + \ldots\ldots + p_{1m}a_m,$$

* Jordan, Essai sur la geometrie à n dimensions. Bull de la Soc. Math. de France, III, *1875*.

perpendicular to the sub-space, *i.e.*, perpendicular to each of the vectors forming the sub-space. Consequently,

$$p_{11}=\left\{\begin{matrix} b_1 a_2 a_3 \ldots a_m \\ a_1 a_2 a_3 \ldots a_m \end{matrix}\right\} \Big/ \left\{\begin{matrix} a_1 a_2 \ldots a_m \\ a_1 a_2 \ldots a_m \end{matrix}\right\},$$

$$p_{12}=\left\{\begin{matrix} a_1 b_1 a_3 \ldots a_m \\ a_1 a_2 a_3 \ldots a_m \end{matrix}\right\} \Big/ \left\{\begin{matrix} a_1 a_2 \ldots a_m \\ a_1 a_2 \ldots a_m \end{matrix}\right\},$$

and so on.

If θ be the angle between b_1 and its projection, then it can be shown

$$\left\{\begin{matrix} b_1 \\ b_1 \end{matrix}\right\} \left\{\begin{matrix} a_1 a_2 \ldots a_m \\ a_1 a_2 \ldots a_m \end{matrix}\right\} \sin^2\theta = \left\{\begin{matrix} b_1 a_1 a_2 \ldots a_m \\ b_1 a_1 a_2 \ldots a_m \end{matrix}\right\},$$

where

$$\left\{\begin{matrix} b_1 \\ b_1 \end{matrix}\right\} = (b/b)_{11} = b_{11}b_{11}+b_{12}b_{12}+\ldots\ldots+b_{1n}b_{1n}$$

$$= \text{sq. of the length of } b_1.$$

11. We next proceed to obtain some more properties of the scalar determinant

$$\left\{\begin{matrix} a_1 \ a_2 \ldots a_m \\ b_1 \ b_2 \ldots b_m \end{matrix}\right\},$$

which we shall represent by Δ_b^a or simply by Δ.

If $j_1, j_2, \ldots j_r$ be a sequence of r distinct integers in natural order, like the sequence $i_1, i_2, \ldots i_r$ defined in Art 5 and $j_{r+1}, j_{r+2}, \ldots j_m$ be the remaining $m-r$ integers also arranged in natural order, then

$$(-1)^{\mathrm{I}+\mathrm{J}} \left\{\begin{matrix} a_{i_1} a_{i_2} \ldots a_{i_r} a_{i_{r+1}} \ldots a_{i_m} \\ b_{j_1} b_{j_2} \ldots b_{j_r} b_{j_{r+1}} \ldots b_{j_m} \end{matrix}\right\} = \Delta, \qquad \ldots \quad (19)$$

where J like I stands for the sum $j_1+j_2+\ldots+j_r$.

In the above, the minor determinants

$$\left\{\begin{matrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ b_{f_1} b_{j_2} \ldots b_{j_r} \end{matrix}\right\} \text{ and } (-1)^{\mathrm{I}+\mathrm{J}} \left\{\begin{matrix} a_{i_{r+1}} \ldots a_{i_m} \\ b_{j_{r+1}} \ldots b_{j_m} \end{matrix}\right\},$$

are said to be *complementary* to one another.

Denoting in particular, the *complement* of $\begin{Bmatrix} a_{i_r} \\ b_{j_s} \end{Bmatrix}$ by $\mu^{i_r}_{j_s}$, we have

$$\begin{Bmatrix} a_{i_r} \\ b_{j_1} \end{Bmatrix} \mu^{i_r}_{j_1} + \begin{Bmatrix} a_{i_r} \\ b_{j_2} \end{Bmatrix} \mu^{i_r}_{j_2} + \ldots\ldots + \begin{Bmatrix} a_{i_r} \\ b_{j_m} \end{Bmatrix} \mu^{i_r}_{j_m} = \Delta,$$

$$\begin{Bmatrix} a_{i_r} \\ b_{j_1} \end{Bmatrix} \mu^{i_s}_{j_1} + \begin{Bmatrix} a_{i_r} \\ b_{j_2} \end{Bmatrix} \mu^{i_s}_{j_2} + \ldots\ldots + \begin{Bmatrix} a_{i_r} \\ b_{j_m} \end{Bmatrix} \mu^{i_s}_{j_m} = 0, \quad r \neq s.$$

The following identity can now be easily established:

$$\begin{Bmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} a_{i_{r+1}} \ldots a_{i_m} \\ b_{j_1} b_{j_2} \ldots b_{j_r} b_{j_{r+1}} \ldots b_{j_m} \end{Bmatrix} \times \begin{vmatrix} \mu^{i_1}_{j_1} & \mu^{i_1}_{j_2} & \ldots & \mu^{i_1}_{j_r} \\ \mu^{i_2}_{j_1} & \mu^{i_2}_{j_2} & \ldots & \mu^{i_2}_{j_r} \\ \ldots & \ldots & & \ldots \\ \mu^{i_r}_{j_1} & \mu^{i_r}_{j_2} & \ldots & \mu^{i_r}_{j_r} \end{vmatrix}$$

$$= \Delta^r . \begin{Bmatrix} a_{i_{r+1}} a_{i_{r+2}} \ldots a_{i_m} \\ b_{j_{r+1}} b_{j_{r+2}} \ldots b_{j_m} \end{Bmatrix}.$$

Using the symbol $\left|\mu^i_j\right|^r_r$ to denote the determinant involving μ, we can rewrite the above in the form

$$\left| \mu^i_j \right|^r_r = \Delta^{r-1}(-1)^{\mathrm{I}+\mathrm{J}} \begin{Bmatrix} a_{i_{r+1}} \ldots a_{i_m} \\ b_{j_{r+1}} \ldots b_{j_m} \end{Bmatrix}$$

$$= \Delta^{r-1} \begin{Bmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ b_{j_1} b_{j_2} \ldots b_{j_r} \end{Bmatrix}_\mu, \qquad \ldots \quad (20)$$

where the suffix μ attached to the minor indicates its complement to be taken.

12. If in the scalar determinant Δ^q_p, we replace $a_{i_1}, a_{i_2}, \ldots a_{i_r}$ by $c_{j_1}, c_{j_2}, \ldots c_{j_r}$ respectively, then the resulting determinant is expressed by the symbol

$$\begin{pmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ c_{j_1} c_{j_2} \ldots c_{j_r} \end{pmatrix} \Delta.$$

Again, if $b_{i_1}\, b_{i_2} \dots b_{i_r}$ be replaced by $c_{j_1}\, c_{j_2} \dots c_{j_r}$ we use the symbol

$$\begin{pmatrix} c_{j_1}\, c_{j_2} \dots c_{j_r} \\ b_{i_1}\, b_{i_2} \dots b_{i_r} \end{pmatrix} \Delta,$$

to represent the determinant.

Let $\begin{pmatrix} a_{i_r} \\ c_{j_s} \end{pmatrix} \Delta$ represent Δ in which a_{i_r} is replaced by c_{j_s}; we have then

$$\begin{pmatrix} a_{i_r} \\ c_{j_s} \end{pmatrix} \Delta = \begin{Bmatrix} c_{j_s} \\ b_1 \end{Bmatrix} \mu_1^{i_r} + \begin{Bmatrix} c_{j_s} \\ b_2 \end{Bmatrix} \mu_2^{i_r} + \dots\dots + \begin{Bmatrix} c_{j_s} \\ b_m \end{Bmatrix} \mu_m^{i_r}.$$

Hence the determinant

$$\begin{vmatrix} \begin{pmatrix} a_{i_1} \\ c_{j_1} \end{pmatrix} \Delta & \begin{pmatrix} a_{i_2} \\ c_{j_1} \end{pmatrix} \Delta \dots\dots & \begin{pmatrix} a_{i_r} \\ c_{j_1} \end{pmatrix} \Delta \\ \begin{pmatrix} a_{i_1} \\ c_{j_2} \end{pmatrix} \Delta & \begin{pmatrix} a_{i_2} \\ c_{j_2} \end{pmatrix} \Delta \dots\dots & \begin{pmatrix} a_{i_r} \\ c_{j_2} \end{pmatrix} \Delta \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ \begin{pmatrix} a_{i_1} \\ c_{j_r} \end{pmatrix} \Delta & \begin{pmatrix} a_{i_2} \\ c_{j_r} \end{pmatrix} \Delta \dots\dots & \begin{pmatrix} a_{i_r} \\ c_{j_r} \end{pmatrix} \Delta \end{vmatrix},$$

for which we use the symbol $\begin{pmatrix} a_i \\ c_j \end{pmatrix}_r^r \Delta$, resolves into the product of two arrays

$$\begin{vmatrix} \begin{Bmatrix} c_{j_1} \\ b_1 \end{Bmatrix} & \begin{Bmatrix} c_{j_1} \\ b_2 \end{Bmatrix} \dots\dots\dots & \begin{Bmatrix} c_{j_1} \\ b_m \end{Bmatrix} \\ \begin{Bmatrix} c_{j_2} \\ b_1 \end{Bmatrix} & \begin{Bmatrix} c_{j_2} \\ b_2 \end{Bmatrix} \dots\dots\dots & \begin{Bmatrix} c_{j_2} \\ b_m \end{Bmatrix} \\ \dots & \dots & \dots \\ \begin{Bmatrix} c_{j_r} \\ b_1 \end{Bmatrix} & \begin{Bmatrix} c_{j_r} \\ b_2 \end{Bmatrix} \dots\dots\dots & \begin{Bmatrix} c_{j_r} \\ b_m \end{Bmatrix} \end{vmatrix} \text{ and } \begin{vmatrix} \mu_1^{i_1} & \mu_2^{i_1} \dots\dots\dots & \mu_m^{i_1} \\ \mu_1^{i_2} & \mu_2^{i_2} \dots\dots\dots & \mu_m^{i_2} \\ \dots & \dots\dots\dots\dots & \dots \\ \dots & \dots\dots\dots\dots & \dots \\ \mu_1^{i_r} & \mu_2^{i_r} \dots\dots\dots & \mu_m^{i_r} \end{vmatrix},$$

each of r rows and m columns.

Using the expression (5) for the product, we have for a typical term in the above

$$\begin{Bmatrix} c_{i_1} & c_{i_2} \ldots\ldots c_{i_r} \\ b_{k_1} & b_{k_2} \ldots\ldots b_{k_r} \end{Bmatrix} |\mu^i_k|^r_r ,$$

where k_1, k_2, ... k_r is one of the $\binom{m}{r}$ combinations of 1, 2,......m taken r at a time in natural order.

By using (20), the above may be written

$$\Delta^{r-1} \begin{Bmatrix} c_{i_1} c_{i_2} \ldots c_{i_r} \\ b_{k_1} b_{k_2} \ldots b_{k_r} \end{Bmatrix} \begin{Bmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ b_{k_1} b_{k_2} \ldots b_{k_r} \end{Bmatrix}_\mu .$$

Now by Laplace's expansion

$$\sum_k \begin{Bmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ b_{k_1} b_{k_2} \ldots b_{k_r} \end{Bmatrix} \begin{Bmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ b_{k_1} b_{k_2} \ldots b_{k_r} \end{Bmatrix}_\mu = \Delta .$$

Hence the product may be expressed in the form

$$\Delta^{r-1} \begin{pmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ c_{j_1} c_{j_2} \ldots c_{j_r} \end{pmatrix} \Delta .$$

$$\text{Thus} \quad \begin{pmatrix} a_i \\ c_j \end{pmatrix}^r_r \Delta = \Delta^{r-1} \begin{pmatrix} a_{i_1} a_{i_2} \ldots a_{i_r} \\ c_{j_1} c_{j_2} \ldots c_{j_r} \end{pmatrix} \Delta . \qquad \ldots \quad (21)$$

13 Let us now revert to Art. 10 and consider projections more fully

Denoting the orthogonal projection of b_i by c_i, we have

$$c_i = p_{i_1} a_1 + p_{i_2} a_2 + \ldots\ldots\ldots + p_{i_m} a_m ,$$

where
$$\Delta_a p_{i_j} = \begin{pmatrix} a_j \\ b_i \end{pmatrix} \Delta_a .$$

In the above Δ_a stands for the scalar determinant

$$\begin{Bmatrix} a_1 \; a_2 \ldots\ldots a_m \\ a_1 \; a_2 \ldots\ldots a_m \end{Bmatrix} .$$

Since $b_i - c_i$ is perpendicular to each of the vectors $a_1, a_2, \ldots a_m$, we have

$$\begin{Bmatrix} c_1 & c_2 \ldots\ldots c_r \\ c_1 & c_2 \ldots\ldots c_r \end{Bmatrix} = \begin{Bmatrix} b_1 & b_2 \ldots\ldots b_r \\ c_1 & c_2 \ldots\ldots c_r \end{Bmatrix},$$

$$\begin{Bmatrix} c_1 & c_2 \ldots\ldots\ldots\ldots c_r \\ a_{i_1} & a_{i_2} \ldots\ldots a_{i_r} \end{Bmatrix} = \begin{Bmatrix} b_1 & b_2 \ldots\ldots\ldots\ldots b_r \\ a_{i_1} & a_{i_2} \ldots\ldots a_{i_r} \end{Bmatrix}, \quad r \leq m \qquad (22)$$

Now by (12),

$$\begin{Bmatrix} b_1 & b_2 \ldots\ldots b_r \\ c_1 & c_2 \ldots\ldots c_r \end{Bmatrix} = \Sigma \mathrm{P}_{i_1 i_2 \ldots i_r} \begin{Bmatrix} a_{i_2} & a_{i_2} \ldots a_{i_r} \\ b_1 & b_2 \ldots\ldots b_r \end{Bmatrix},$$

where $$(\Delta_a)^r \mathrm{P}_{i_1 i_2 \ldots i_r} = \begin{pmatrix} a_i \\ b \end{pmatrix}_r^r \Delta_a .$$

But by (21),

$$\begin{pmatrix} a_i \\ b \end{pmatrix}_r^r \Delta_a = (\Delta_a)^{r-1} \begin{pmatrix} a_{i_1} & a_{i_2} \ldots\ldots a_{i_r} \\ b_1 & b_2 \ldots\ldots\ldots b_r \end{pmatrix} \Delta_a .$$

Therefore

$$\Delta_a \begin{Bmatrix} b_1 & b_2 \ldots b_r \\ c_1 & c_2 \ldots c_r \end{Bmatrix} = \sum \begin{pmatrix} a_{i_1} & a_{i_2} \ldots a_{i_r} \\ b_1 & b_2 \ldots\ldots b_r \end{pmatrix} \Delta_a \cdot \begin{Bmatrix} a_{i_1} & a_{i_2} \ldots a_{i_r} \\ b_1 & b_2 \ldots\ldots b_r \end{Bmatrix} \ldots \quad (23)$$

When $r = m$, we have

$$\Delta_a \begin{Bmatrix} b_1 & b_2 \ldots b_m \\ c_1 & c_2 \ldots c_m \end{Bmatrix} = \begin{Bmatrix} a_1 & a_2 \ldots a_m \\ b_1 & b_2 \ldots b_m \end{Bmatrix} \begin{Bmatrix} a_1 & a_2 \ldots a_m \\ b_1 & b_2 \ldots b_m \end{Bmatrix}. \quad \ldots \quad (24)$$

Let us consider r-dimensional sub-space formed by the vectors b_1, $b_2, \ldots b_r$, then the expression

$$\frac{\begin{Bmatrix} b_1 & b_2 \ldots b_r \\ c_1 & c_2 \ldots c_r \end{Bmatrix}}{\sqrt{\begin{Bmatrix} b_1 & b_2 \ldots b_r \\ b_1 & b_2 \ldots b_r \end{Bmatrix}} \sqrt{\begin{Bmatrix} c_1 & c_2 \ldots c_r \\ c_1 & c_2 \ldots c_r \end{Bmatrix}}} = \frac{\sqrt{\begin{Bmatrix} c_1 & c_2 \ldots c_r \\ c_1 & c_2 \ldots c_r \end{Bmatrix}}}{\sqrt{\begin{Bmatrix} b_1 & b_2 \ldots b_r \\ b_1 & b_2 \ldots b_r \end{Bmatrix}}},$$

by (22).

The numerator in the right-hand expression represents the content of the 'projected' sub-space of r dimensions in the m-dimensional sub-space, formed by the vectors $a_1, a_2, \ldots a_m$. The above measures the angle between sub-spaces.

From (24), we have

$$\begin{Bmatrix} a_1 & a_2 \ldots\ldots a_m \\ b_1 & b_2 \ldots\ldots b_m \end{Bmatrix} = \sqrt{\begin{Bmatrix} a_1 & a_2 \ldots\ldots a_m \\ a_1 & a_2 \ldots\ldots a_m \end{Bmatrix}} \sqrt{\begin{Bmatrix} c_1 & c_2 \ldots\ldots c_m \\ c_1 & c_2 \ldots\ldots c_m \end{Bmatrix}},$$

which gives an interpretation of the scalar determinant Δ^a_b.

14. We conclude this paper by remarking that a scalar determinant can always be expressed as a product of scalar determinants.

Let Δ^a_b denote the scalar determinant

$$\begin{Bmatrix} a_1 & a_2 \ldots\ldots a_m \\ b_1 & b_2 \ldots\ldots b_m \end{Bmatrix}.$$

Fixing upon a minor

$$\begin{Bmatrix} a_1 & a_2 \ldots\ldots a_r \\ b_1 & b_2 \ldots\ldots b_r \end{Bmatrix},$$

denoted by $\Delta^{a_r}_{b_r}$, let us replace each of the remaining vectors a_s in Δ^a_b by the vector

$$a'_s = a_s - \sum_{i=1}^{r} a_i \begin{pmatrix} a_i \\ a_s \end{pmatrix} \Delta^{a_r}_{b_r} \Big/ \Delta^{a_r}_{b_r} \quad \ldots \quad \ldots \quad (25)$$

This operation does not change the value of Δ^a_b, which appears in the new form

$$\begin{Bmatrix} a_1 & a_2 \ldots a_r & a'_{r+1} & a'_{r+2} \ldots a'_m \\ b_1 & b_2 \ldots b_r & b_{r+1} & b_{r+2} \ldots b_m \end{Bmatrix} \quad \ldots \quad \ldots \quad (26)$$

Now it is easy to see that a'_s is orthogonal to each of b_k $(k \leq r)$, for the scalar determinant

$$\begin{Bmatrix} a_s & a_1 & a_2 \ldots a_r \\ b_k & b_1 & b_2 \ldots b_r \end{Bmatrix},$$

identically vanishes if $k \leq r$.

Thus we have expressed Δ^a_b in the product form

$$\begin{Bmatrix} a_1 & a_2 \ldots a_r \\ b_1 & b_2 \ldots b_r \end{Bmatrix} \begin{Bmatrix} a'_{r+1} & a'_{r+2} \ldots a'_m \\ b_{r+1} & b_{r+2} \ldots b_m \end{Bmatrix}. \quad \ldots \quad (27)$$

Interchanging a_i and b_i we notice, further,

$$\begin{Bmatrix} a'_{r+1} & a'_{r+2} \ldots a'_m \\ b_{r+1} & b_{r+2} \ldots b_m \end{Bmatrix} = \begin{Bmatrix} b'_{r+1} & b'_{r+2} \ldots b'_m \\ a_{r+1} & a_{r+2} \ldots a_m \end{Bmatrix}. \quad \ldots \quad (28)$$

A repetition of the formula (27) leads to what may be called the canonical form of a scalar determinant.

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2

ON A CLASS OF INTEGRAL EQUATIONS

BY

MAURICE DE DUFFAHEL.

1. *Introductory.*

In a paper which appeared in the *Bulletin de Mathematiques Supérieures* ; supplement, Vol. XXXII, Session 1932-33, * I showed how the application of Laplace's transformation to certain linear differential equations enables us to solve some homogeneous integral equations of the first and the second kinds, and I obtained by an extension of this method, the solution of integral equations whose nucleus is of the form $f(zt)$ or $e^{f(z)f(t)}$.

I propose now to show how this method furnishes the solution of an infinite number of homogeneous integral equations of the first and the second kinds where the nucleus $f(z,t)$ satisfies a partial differential equation with respect to z and t, of a rather general form ; and I shall then make some remarks on these integral equations.

2. *Exposition of the method.*

Z and S being functions of z, and h an arbitrary constant, let us consider the linear differential equation of the first order

$$Z\frac{dy}{dz}+(S+h)y = 0. \qquad \dots (1)$$

Its solution is

$$y(z) = C\, e^{-\int \frac{s+h}{Z} dz}$$

Let us now make, in the equation (1), the following change of variable

$$y(z) = \int^{b} f(z,t)v(t)dt,$$

* References to this paper will be made in the present one under the symbol I, followed by the indication of the paragraph.

where a and b are two constants, conveniently chosen, and where $f(z,t)$ is a function of z and t, which satisfies the partial differential equation of the first order

$$Z\frac{\partial f}{\partial z}+T\frac{\partial f}{\partial z}+(S+\theta)f=0. \qquad \dots \quad (2)$$

In this equation Z and S are functions of z already spoken of, and T and θ are functions of t. It is evident that neither Z nor T can vanish. All our reasoning supposes that essential condition.

We have

$$\frac{dy}{dz}=\int_a^b\frac{\partial f}{\partial z}v\,dt,$$

and equation (1) becomes

$$\int_a^b v(t)dt\left[Z\frac{\partial f}{\partial z}+Sf+hf\right]=0. \qquad \dots \quad (3)$$

But using equation (2), this becomes

$$\int_a^b v(t)dt\left[-T\frac{\partial f}{\partial z}-\theta f+hf\right]=0.$$

Now we can write, by integrating by parts,

$$\int_a^b\frac{\partial f}{\partial z}Tv(t)dt=\Big[fTv\Big]_a^b-\int_a^b f(Tv'+T'v)\,dt.$$

If then we choose the constants a and b such that

$$[fTv]_a^b=0,$$

and the suitable value of v is given by the linear differential equation,

$$Tv'+v(T'-\theta+h)=0, \quad \dots \qquad \dots \quad (4)$$

where t is the independent variable.

The solution is

$$v(t)=\frac{C}{T}e^{\int\frac{\theta-h}{T}dt},$$

and we find then the homogeneous integral equation of the first kind

$$e^{-\int \frac{s+h}{Z}} = \lambda_h \int_a^b f(z,t)\, e^{\int \frac{\theta-h}{T} dt} \frac{dt}{T} \qquad \dots \quad (5)$$

In this equation a and b must be determined as we have explained above ; we can take two constants, independent of z, such that

$$f(z,a) \quad = \quad f(z,b) \quad = \quad o.$$

The value of λ, which is also to be determined, will be easily obtained by giving to z a known value, such as 0 or 1. Its value changes with the constant h.

It may be understood that we are thus led to a very general class of integral equations. Depending upon a partial differential equation, the nuclei will always contain an arbitrary function ; but generally, they will be unsymmetrical.

The two nuclei studied in the former paper $f(zt)$ and $e^{f(z)f(t)}$ (I, § 3-5), are mere particular cases of those we are now speaking of.

3. *Case in which the integral equation is of the second kind.*

As was pointed out in § 2 of our other paper, the integral equation will be of the second kind when the equation (4) turns out to be precisely the same as (1) ; this occurs if we have between the four functions Z, S, T, θ (in which we suppose a single variable, say u, has been written instead of z and t), and k, the relation

$$\frac{S+h}{Z} = \frac{h+T'-\theta}{T} \qquad \dots \quad (6)$$

Very often, for a given function $f(z,t)$, this condition will be fulfilled only for a particular value of h. For instance, with the nucleus $f(zt)$, it is fulfilled only for $h=\frac{1}{2}$; if we give to h any other value, we shall obtain an integral equation of the first kind.

4. *Examples and applications.*

A. Let us consider, as a nucleus, the function

$$f(z,t) \quad = \quad \frac{1}{z+t}\phi(\frac{z}{t}),$$

ϕ being an arbitrary function.

It satisfies the partial differential equation

$$z\frac{\partial f}{\partial z}+t\frac{\partial f}{\partial t}+f = 0,$$

which is of the form in question ; and in that case, as we have

$$Z = z, \quad T = t, \quad S = 1, \quad \theta = 0,$$

the corresponding integral equation will be of the second kind.

It is easily seen that the solution is z^k, k being a constant. So we have the integral equation of the second kind

$$z^k = \lambda_k \int_a^b \phi\left(\frac{z}{t}\right) \frac{1}{z+t}\, t^k\, dt.$$

If, for $t = o$, we have $\phi\left(\frac{z}{t}\right)=o$, and if, for $t = +\infty$,

$\phi\left(\frac{z}{t}\right) \neq 0$, we can take 0 and $+\infty$ for the limits a and b.

This will be the case, for instance, if we take

$$\phi\left(\frac{z}{t}\right) = e^{-\frac{z}{k}}.$$

The value of λ_k will be obtained by making $z = 1$, and so

$$\frac{1}{\lambda_k} = \int_0^\infty \varphi\left(\frac{1}{t}\right) \frac{t^k}{t+1}\, dt.$$

We are now able to solve the equation of the first kind

$$F(z) = \int_0^\infty \phi\left(\frac{z}{t}\right)\frac{1}{z+t}\, \Phi(t)\, dt,$$

where F is known and Φ unknown: for, let us expand F in a Laurent series for points z within an annulus whose centre is at the origin:

$$F(z) = z^k[a_0+a_1 z+a_2 z^2+\ldots\ldots+b_1 z^{-1}+b_2 z^{-2}+\ldots\ldots],$$

we have for the unknown function $\Phi(t)$,

$$\Phi(t) = t^k[\lambda_k a_0+\lambda_{k+1}\, a_1 t+\lambda_{k+2} a_2 t^2+\ldots+\lambda_{k-1}\, b_1 t^{-1}$$
$$+\lambda_{k-2} b_2 t^{-2}+\ldots],$$

the λ's being given by the above formula.

B. The function

$$f_1(z,t) = \phi\left(\frac{z}{t}\right)\frac{1}{z-t},$$

satisfying the same partial differential equation, the above results remain unchanged for integral equations with this function f_1 as nucleus.

C. Again, the function

$$f(z,t) = \phi\left(\frac{z}{t}\right)\frac{1}{z^n \pm t^n},$$

which satisfies the partial differential equation

$$z\frac{\partial f}{\partial z} + t\frac{\partial f}{\partial t} + nf = 0,$$

leads to the integral equation of the first kind,

$$z^k = \lambda_k \int_0^{\infty} \phi\left(\frac{z}{t}\right)\frac{1}{z^n \pm t^n}\, t^{n+k-1}\, dt.$$

As above, we can deduce from this equation the solution of the integral equation of the first kind,

$$F(z) = \int \phi\left(\frac{z}{t}\right)\frac{1}{z^n \pm t^n}\,\Phi(t)\, dt,$$

by the process of developing $F(z)$ in a Laurent's series of powers of z.

D. Another example of integral equations of the first kind obtained by our method is the following:

$$e^{-\frac{1}{2}z^2} = \lambda \int_{-\infty}^{\infty} \phi(z+t)\, e^{zt}\, e^{\frac{1}{2}t^2}\, dt,$$

the functions ϕ being such as

$$\lim_{u=\pm\infty} \phi(u)\, e^{u} = 0.$$

5. *Change of Variable.*

A great number of new integral equations (generally of the first kind) can be deduced from known ones by change of variable. For, if

we know the function $v(t)$ which satisfies the integral equation

$$y(z) = \int f(z,t)\, v(t)\, dt,$$

we can at once write the solution of the integral equation

$$y_1(z) = y\,(\zeta) = \int f(\zeta,\, \theta)\, v_1(t)\, dt,$$

where ζ is a function of z, θ a function of t: it appears clearly that

$$v_1(t) = v(\theta)\frac{d\theta}{dt}\,.$$

It is by using the change of variable that it is possible to deduce an integral equation of the second kind from one of the first kind (see an example of this in I. § 7).

It may be seen that the equation obtained above in example C can be deduced from the equations in examples A and B by writing

$$\zeta^n = z, \quad \theta^n = t,$$

and also the integral equation

$$z^k = \lambda_k \int \phi\left(\frac{z}{t}\right) t^{k-1}\, dt,$$

which is a particular case of example C, with $n=0$, can be deduced from the following (I., § 4)

$$z^k = \lambda_k \int \phi\,(zt)\, t^{-1-k}\, dt,$$

by changing t into $\dfrac{1}{t}$.

Starting from this last formula, let us put, instead of t, a function $g\,(t)$. We have

$$z^k = \lambda_k \int \phi\,[z\, g(t)][g(t)]^{-1-k}\, g'(t)\, dt,$$

and this gives us the solution of the integral equation of the first kind

$$\mathrm{F}\,(z) = \int \phi\,[z\, g(t)]\, \Phi(t)\, dt,$$

which solution will be obtained by the process of developing F (z) in eries, as we did in the preceding paragraph.

For instance, let us take

$$g(t) = \log t ;$$

we have

$$z^k = \lambda_k \int \phi(z \log t) (\log t)^{-1-k} \frac{dt}{t},$$

which gives us the solution of integral equations of first kind with the nucleus ϕ $(z\log t)$ or, what is the same, ϕ (t^z). This result can also be obtained by our direct method, the function

$$f(z, t) = \phi(t^z)$$

satisfying the partial differential equation

$$z \frac{\partial f}{\partial z} - t \log t \frac{\partial f}{\partial t} = 0.$$

A similar change of variables gives us the integral equation

$$[f_1(z)]^k = \lambda_k \int \phi [f_1(z) f_2(t)] [f_2(t)]^{-1-k} f'_2(t)\, dt,$$

and gives us a method of solving an integral equation of the first kind, with this nucleus, when the known function of z can be developed in a series of powers of $f_1(z)$.

6. *Extension.*

It will be easily understood that our method can be extended to nuclei $f(z,t)$ satisfying a partial differential equation of the second order, of suitable form. The differential equations for y and v will in this case be of the second order ; and we shall then obtain a relation such as

$$c_1 y_1 + c_2 y_2 = \lambda \int f(z, t) [c'_1 v_1 + c'_2 v_2]\, dt,$$

the y's and v's being particular solutions of their respective differential equations. This is an integral equation of the first kind and one has to determine the suitable values of the c's. This case is much

more complicated than the case of the first order, and naturally, so also would be those obtained by extension to the third, fourth,......nth orders.

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3

Plane Strain in an Infinite Plate with an Elliptic Hole

BY

S. Ghosh.

The object of the present paper is to give a solution of the problem of plane strain in an infinite plate with an elliptic hole, when the surface tractions on the elliptic boundary, are prescribed. The corresponding case of given surface displacements, has already been treated by Love,[1] without the use of the stress function χ. In the present case, it has been found advantageous to introduce the stress function χ, whose form in elliptic co-ordinates, is well-known.

The co-ordinates.

Let

$$x = c \cosh \alpha \cos \beta, \quad y = c \sinh \alpha \sin \beta, \tag{1}$$

so that

$$\frac{1}{h^2} = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial x}{\partial \beta}\right)^2$$

$$= \frac{c^2}{2}(\cosh 2\alpha - \cos 2\beta). \tag{2}$$

The curves α = constant, are confocal ellipses and the curves β = constant, are confocal hyperbolas.

In a previous paper,[2] it has been found that the stress components and the displacements are given by

$$\left.\begin{aligned}
\frac{\widehat{\alpha\alpha}}{h^4} &= \frac{c^2}{2}\left[(\cosh 2\alpha - \cos 2\beta)\frac{\partial^2 \chi}{\partial \beta^2} + \sinh 2\alpha \frac{\partial \chi}{\partial \alpha} - \sin 2\beta \frac{\partial \chi}{\partial \beta}\right], \\
\frac{\widehat{\beta\beta}}{h^4} &= \frac{c^2}{2}\left[(\cosh 2\alpha - \cos 2\beta)\frac{\partial^2 \chi}{\partial \alpha^2} - \sinh 2\alpha \frac{\partial \chi}{\partial \alpha} + \sin 2\beta \frac{\partial \chi}{\partial \beta}\right], \\
\frac{\widehat{\alpha\beta}}{h^4} &= \frac{c^2}{2}\left[-(\cosh 2\alpha - \cos 2\beta)\frac{\partial^2 \chi}{\partial \alpha \partial \beta} + \sinh 2\alpha \frac{\partial \chi}{\partial \beta} + \sin 2\beta \frac{\partial \chi}{\partial \alpha}\right],
\end{aligned}\right\} \tag{3}$$

[1] Treatise on the Theory of Elasticity (1st. Ed.), Vol. I, *340-44*.

[2] "On the solution of the equations of elastic equilibrium suitable for elliptic boundaries."—Trans. American Math. Soc., 32, No. 1, *47-60*.

and

$$\left.\begin{aligned} \frac{2\mu u}{h} &= -\frac{\partial \chi}{\partial \alpha} + \frac{1}{h^2}\frac{\partial P}{\partial \beta}, \\ \frac{2\mu v}{h} &= -\frac{\partial \chi}{\partial \beta} + \frac{1}{h^2}\frac{\partial P}{\partial \alpha}, \end{aligned}\right\} \tag{4}$$

where

$$\nabla_1^2 P = 0,$$

and

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{h^2}\frac{\partial P}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{h^2}\frac{\partial P}{\partial \alpha}\right) = \frac{\lambda+2\mu}{\lambda+\mu}\left(\frac{\partial^2 \chi}{\partial \alpha^2} + \frac{\partial^2 \chi}{\partial \beta^2}\right). \tag{5}$$

The method of calculating $(1/h^2)(\partial P/\partial \alpha)$, $(1/h^2)(\partial P/\partial \beta)$ has already been given in that paper.

The stress function χ satisfies the equation,

$$\nabla_1^4 \chi = 0, \tag{6}$$

whose solution in elliptic coordinates, has been found in the paper cited before. The terms in χ, which give rise to many-valued displacements, have been been found to be

$$\alpha(\cosh 2\alpha + \cos 2\beta), \quad \alpha \cosh \alpha \cos \beta, \quad \alpha \sinh \alpha \sin \beta.$$

It can also be verified that

$$\beta \cosh \alpha \cos \beta, \quad \beta \sinh \alpha \sin \beta,$$

are solutions of (6) and give rise to single-valued stresses. It will subsequently be shown that the corresponding displacements are many-valued.

Many-valued displacements.[1]

If

$$\chi = \alpha(\cosh 2\alpha + \cos 2\beta),$$

it has already been found in the paper previously referred to, that

$$\left.\begin{aligned} \frac{1}{h^2}\frac{\partial P}{\partial \alpha} &= \frac{2(\lambda+2\mu)}{\lambda+\mu}(\beta \sinh 2\alpha - \alpha \sin 2\beta), \\ \frac{1}{h^2}\frac{\partial P}{\partial \beta} &= \frac{2(\lambda+2\mu)}{\lambda+\mu}(\beta \sin 2\beta + \alpha \sinh 2\alpha), \end{aligned}\right\} \tag{7}$$

[1] The many-valued displacements have been obtained in a slightly different manner by Timpe, Math. Zeit. 17, *189*.

so that

$$\left.\begin{aligned}\frac{2\mu u}{h} &= -(\cosh 2\alpha + \cos 2\beta) + \frac{2(\lambda+2\mu)}{\lambda+\mu}\,\beta \sin 2\beta + \frac{2\mu}{\lambda+\mu}\,\alpha \sinh 2\alpha,\\ \frac{2\mu v}{h} &= \frac{2(\lambda+2\mu)}{\lambda+\mu}\,\beta \sinh 2\alpha - \frac{2\mu}{\lambda+\mu}\,\alpha \sin 2\beta.\end{aligned}\right\} \quad (8)$$

If

$$\chi = \alpha \cosh \alpha \cos \beta,$$

it has been shown in the same paper, that

$$\left.\begin{aligned}\frac{1}{h^2}\frac{\partial P}{\partial \alpha} &= A'\,(\beta \sinh \alpha \cos \beta - \alpha \cosh \alpha \sin \beta),\\ \frac{1}{h^2}\frac{\partial P}{\partial \beta} &= A'\,(\beta \cosh \alpha \sin \beta + \alpha \sinh \alpha \cos \beta),\end{aligned}\right\} \quad (9)$$

and A' is found from the equation (5), which gives

$$A' = \frac{\lambda+2\mu}{\lambda+\mu}.$$

Hence

$$\left.\begin{aligned}\frac{2\mu u}{h} &= -\cosh \alpha \cos \beta + \frac{\lambda+2\mu}{\lambda+\mu}\,\beta \cosh \alpha \sin \beta + \frac{\mu}{\lambda+\mu}\,\alpha \sinh \alpha \cos \beta,\\ \frac{2\mu v}{h} &= \frac{\lambda+2\mu}{\lambda+\mu}\,\beta \sinh \alpha \cos \beta - \frac{\mu}{\lambda+\mu}\,\alpha \cosh \alpha \sin \beta.\end{aligned}\right\} \quad (10)$$

If

$$\chi = \alpha \sinh \alpha \sin \beta,$$

it has also been found in the same paper, that

$$\left.\begin{aligned}\frac{1}{h^2}\frac{\partial P}{\partial \alpha} &= B'\,(\alpha \sinh \alpha \cos \beta + \beta \cosh \alpha \sin \beta),\\ \frac{1}{h^2}\frac{\partial P}{\partial \beta} &= B'\,(\alpha \cosh \alpha \sin \beta - \beta \sinh \alpha \cos \beta),\end{aligned}\right\} \quad (11)$$

where B' is found from (5) to be equal to

$$\frac{\lambda+2\mu}{\lambda+\mu}.$$

Hence

$$\left.\begin{aligned}\frac{2\mu u}{h} &= -\sinh\alpha\sin\beta - \frac{\lambda+2\mu}{\lambda+\mu}\beta\sinh\alpha\cos\beta + \frac{\mu}{\lambda+\mu}\alpha\cosh\alpha\sin\beta,\\ \frac{2\mu v}{h} &= \frac{\lambda+2\mu}{\lambda+\mu}\beta\cosh\alpha\sin\beta + \frac{\mu}{\lambda+\mu}\alpha\sinh\alpha\cos\beta.\end{aligned}\right\} \quad (12)$$

If
$$\chi = \beta\cosh\alpha\cos\beta,$$

it can easily be shown that

$$\left.\begin{aligned}\frac{1}{h^2}\frac{\partial P}{\partial\alpha} &= -\frac{\lambda+2\mu}{\lambda+\mu}(\alpha\sinh\alpha\cos\beta+\beta\cosh\alpha\sin\beta),\\ \frac{1}{h^2}\frac{\partial P}{\partial\beta} &= -\frac{\lambda+2\mu}{\lambda+\mu}(\alpha\cosh\alpha\sin\beta-\beta\sinh\alpha\cos\beta).\end{aligned}\right\} \quad (13)$$

Hence

$$\left.\begin{aligned}\frac{2\mu u}{h} &= -\frac{\lambda+2\mu}{\lambda+\mu}\alpha\cosh\alpha\sin\beta + \frac{\mu}{\lambda+\mu}\beta\sinh\alpha\cos\beta,\\ \frac{2\mu v}{h} &= -\cosh\alpha\cos\beta - \frac{\lambda+2\mu}{\lambda+\mu}\alpha\sinh\alpha\cos\beta - \frac{\mu}{\lambda+\mu}\beta\cosh\alpha\sin\beta.\end{aligned}\right\} \quad (14)$$

Finally, when
$$\chi = \beta\sinh\alpha\sin\beta,$$

we can show that

$$\left.\begin{aligned}\frac{1}{h^2}\frac{\partial P}{\partial\alpha} &= \frac{\lambda+2\mu}{\lambda+\mu}(\beta\sinh\alpha\cos\beta-\alpha\cosh\alpha\sin\beta),\\ \frac{1}{h^2}\frac{\partial P}{\partial\beta} &= \frac{\lambda+2\mu}{\lambda+\mu}(\beta\cosh\alpha\sin\beta+\alpha\sinh\alpha\cos\beta).\end{aligned}\right\} \quad (15)$$

Hence

$$\left.\begin{aligned}\frac{2\mu u}{h} &= \frac{\lambda+2\mu}{\lambda+\mu}\alpha\sinh\alpha\cos\beta + \frac{\mu}{\lambda+\mu}\beta\cosh\alpha\sin\beta,\\ \frac{2\mu v}{h} &= -\sinh\alpha\sin\beta\\ &\quad -\frac{\lambda+2\mu}{\lambda+\mu}\alpha\cosh\alpha\sin\beta + \frac{\mu}{\lambda+\mu}\beta\sinh\alpha\cos\beta.\end{aligned}\right\} \quad (16)$$

The displacements (8), (10), (12), (14) and (16) are obviously many-valued displacements, but by suitable combinations of (10) with (16) and (12) with (14), we can get single-valued displacements.

Comparing (10) with (16) and (12) with (14), we find that if we take

$$\chi = A\left(\alpha \cosh \alpha \cos \beta - \frac{\lambda+2\mu}{\mu} \beta \sinh \alpha \sin \beta\right)$$

$$+ B\left(\alpha \sinh \alpha \sin \beta + \frac{\lambda+2\mu}{\mu} \beta \cosh \alpha \cos \beta\right), \tag{17}$$

the displacements produced are obviously single-valued.

It can easily be verified that the stresses calculated from (17) are single-valued.

Stress Function.

Thus the solution of (6), which gives rise to single-valued displacements only, is obtained by considering the value of χ given in (17) with the value of χ given in the paper already mentioned. Separating the many-valued terms, we write the stress function in two parts χ, χ'. Then

$$\chi' = A\left(\alpha \cosh \alpha \cos \beta - \frac{\lambda+2\mu}{\mu} \beta \sinh \alpha \sin \beta\right)$$

$$+ B\left(\alpha \sinh \alpha \sin \beta + \frac{\lambda+2\mu}{\mu} \beta \cosh \alpha \cos \beta\right)$$

$$+ C\beta, \tag{18}$$

and

$$\chi = \phi_0 + \sum_{n=1}^{\infty} \phi_n \cos n\beta + \sum_{n=1}^{\infty} \psi_n \sin n\beta, \tag{19}$$

where

$$\left.\begin{aligned}
&\phi_0 = a_0 \cosh 2\alpha + b_0 \sinh 2\alpha + b'_0 \alpha, \\
&\phi_1 = a_1 \cosh 3\alpha + b_1 \sinh 3\alpha + b'_1 \sinh \alpha, \\
&\phi_2 = a_0 + a_2 \cosh 4\alpha + b_2 \sinh 4\alpha + a'_2 \cosh 2\alpha + b'_2 \sinh 2\alpha, \\
&\text{and for } n \geqq 3, \\
&\phi_n = a_{n-2} \cosh (n-2)\alpha + b_{n-2} \sinh (n-2)\alpha \\
&\qquad + a_n \cosh (n+2)\alpha + b_n \sinh (n+2)\alpha + a'_n \cosh n\alpha \\
&\qquad\qquad + b'_n \sinh n\alpha,
\end{aligned}\right\} \tag{19 a}$$

and

$$\left.\begin{aligned}\psi_1&=c_1\cosh 3\alpha+d_1\sinh 3\alpha+c'_1\cosh\alpha,\\ \psi_2&=c_0+c_2\cosh 4\alpha+d_2\sinh 4\alpha+c'_2\cosh 2\alpha\\ &\qquad+d'_2\sinh 2\alpha,\\ &\text{and for } n\geqq 3,\\ \psi_n&=c_{n-2}\cosh(n-2)\alpha+d_{n-2}\sinh(n-2)\alpha\\ &\quad+c_n\cosh(n+2)\alpha+d_n\sinh(n+2)\alpha+c'_n\cosh n\alpha\\ &\qquad+d'_n\sinh n\alpha.\end{aligned}\right\}\quad(19\ b$$

The terms $a'_1\cosh\alpha$ and $d'_1\sinh\alpha$ have been omitted from ϕ_1 an ψ_1 respectively, for with these terms, we have

$$\chi=a'_1\,x+d'_1\,y,$$

which gives only rigid-body displacements.

We will now show that the terms in (18) correspond to resulta forces and couples applied over the boundaries.

If (r,θ) be the polar co-ordinates of the point (α,β), then

$$r^2=\frac{c^2}{2}(\cosh 2\alpha+\cos 2\beta),$$

so that

$$\log r=\tfrac{1}{2}\log\frac{c^2}{4}+\tfrac{1}{2}\log 2\,(\cosh 2\alpha+\cos 2\beta),$$

$$=\log\frac{c}{2}+\alpha+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-2n\alpha}\cos 2n\beta.$$

Also, since $\log r$ and θ are conjugate harmonic functions,

$$\theta=\beta+\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-2n\alpha}\sin 2n\beta.$$

For pressure radiating uniformly from the origin,

$$\chi=\log r$$

$$=\log\frac{c}{2}+\alpha+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-2n\alpha}\cos 2n\beta.\quad($$

For a couple of moment M at the origin, we have

$$\chi = -\frac{M}{2\pi}\theta$$

$$= -\frac{M}{2\pi}\beta + \frac{M}{2\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-2n\alpha}\sin 2n\beta. \tag{21}$$

For a force X at the origin along the x-axis, we have,

$$\chi = -\frac{X}{2\pi}\left\{y\theta - \frac{\mu}{\lambda+2\mu}x\log r\right\}$$

$$= \frac{\mu c X}{2\pi(\lambda+2\mu)}\left\{\alpha\cosh\alpha\cos\beta - \frac{\lambda+2\mu}{\mu}\beta\sinh\alpha\sin\beta\right\}$$

$$+ \text{ terms which are biharmonic.} \tag{22}$$

For a force Y at the origin along the y-axis, we have

$$\chi = \frac{Y}{2\pi}\left\{x\theta + \frac{\mu}{\lambda+2\mu}y\log r\right\}$$

$$= \frac{\mu c Y}{2\pi(\lambda+2\mu)}\left\{\alpha\sinh\alpha\sin\beta + \frac{\lambda+2\mu}{\mu}\beta\cosh\alpha\cos\beta\right\}$$

$$+ \text{ terms which are biharmonic.} \tag{23}$$

Comparing (21), (22) and (23) with (18), we find that χ' gives rise to resultant forces and couples on each boundary and can therefore be omitted, when we consider the tractions on each boundary to be in equilibrium.

The stress components $\widehat{\alpha\alpha}$, $\widehat{\alpha\beta}$ calculated from (3) with the stress function (19), are given by

$$\frac{2}{c^2}\cdot\frac{\widehat{\alpha\alpha}}{h^4} = (\phi'_0\sinh 2\alpha + 3\phi_2)$$

$$+(-\phi_1\cosh 2\alpha + \phi'_1\sinh 2\alpha + \phi_1 + 6\phi_3)\cos\beta$$

$$+(-\psi_1\cosh 2\alpha + \psi'_1\sinh 2\alpha - \psi_1 + 6\psi_3)\sin\beta$$

$$+\sum_{n=2}^{\infty}[\{-n^2\phi_n\cosh 2\alpha + \phi'_n\sinh 2\alpha$$

$$+\tfrac{1}{2}(n+2)(n+3)\phi_{n+2} + \tfrac{1}{2}(n-2)(n-3)\phi_{n-2}\}\cos n\beta$$

$$+\{-n^2\psi_n\cosh 2\alpha + \psi'_n\sinh 2\alpha$$

$$+\tfrac{1}{2}(n+2)(n+3)\psi_{n+2} + \tfrac{1}{2}(n-2)(n-3)\psi_{n-2}\}\sin n\beta], \tag{24}$$

$$\frac{2}{c^2}\cdot\frac{\widehat{\alpha\beta}}{h^4} = \tfrac{3}{2}\psi'_2 + (\phi'_1\cosh 2\alpha - \phi_1\sinh 2\alpha + \phi'_1 - 2\phi'_3)\sin\beta$$
$$+ (-\psi'_1\cosh 2\alpha + \psi_1\sinh 2\alpha + \psi'_1 + 2\psi'_3)\cos\beta$$
$$+ (\phi'_0 + 2\phi'_2\cosh 2\alpha - 2\phi_2\sinh 2\alpha - \tfrac{5}{2}\phi'_4)\sin 2\beta$$
$$+ (-2\psi'_2\cosh 2\alpha + 2\psi_2\sinh 2\alpha + \tfrac{5}{2}\psi'_4)\cos 2\beta$$
$$+ \sum_{n=3}^{\infty}\Big[\{-\tfrac{1}{2}(n-3)\phi'_{n-2} + n\phi'_n\cosh 2\alpha - n\phi_n\sinh 2\alpha$$
$$-\tfrac{1}{2}(n+3)\phi'_{n+2}\}\sin n\beta$$
$$+\{\tfrac{1}{2}(n-3)\psi'_{n-2} - n\psi'_n\cosh 2\alpha + n\psi_n\sinh 2\alpha$$
$$+\tfrac{1}{2}(n+3)\psi'_{n+2}\}\cos n\beta\Big] \tag{25}$$

Surface Tractions.

Suppose that an infinite plate is bounded internally by the ellipse $\alpha=\alpha_1$. Let the tractions across the boundary $\alpha=\alpha_1$, be given by

$$\left.\begin{aligned}\widehat{\alpha\alpha} &= A_0 + \sum_{m=1}^{\infty}(A_m\cos m\beta + B_m\sin m\beta)\\ \widehat{\alpha\beta} &= C_0 + \sum_{m=1}^{\infty}(C_m\cos m\beta + D_m\sin m\beta)\end{aligned}\right\}. \tag{26}$$

These tractions reduce to forces X, Y and couple L at the origin, where

$$X = \int_0^{2\pi}\left(\widehat{\alpha\alpha}\,\frac{\partial x}{\partial\alpha} - \widehat{\alpha\beta}\,\frac{\partial y}{\partial\alpha}\right)d\beta$$

$$= \pi c\,(A_1\sinh\alpha_1 - D_1\cosh\alpha_1),$$

$$Y = \int_0^{2\pi}\left(\widehat{\alpha\alpha}\,\frac{\partial y}{\partial\alpha} + \widehat{\alpha\beta}\,\frac{\partial x}{\partial\alpha}\right)d\beta$$

$$= \pi c\,(B_1\cosh\alpha_1 + C_1\sinh\alpha_1),$$

and

$$L = \int_0^{2\pi}\widehat{\alpha\alpha}\left(x\frac{\partial y}{\partial\alpha} - y\frac{\partial x}{\partial\alpha}\right)d\beta + \int_0^{2\pi}\widehat{\alpha\beta}\left(x\frac{\partial x}{\partial\alpha} + y\frac{\partial y}{\partial\alpha}\right)d\beta$$

$$= \frac{\pi c^2}{2}(B_2 + 2\,C_0\sinh 2\alpha_1),$$

since when $\alpha=\alpha_1$, we have

$$\frac{\partial x}{\partial \alpha} = c \sinh \alpha_1 \cos \beta, \qquad \frac{\partial x}{\partial \beta} = c \cosh \alpha_1 \sin \beta.$$

If the tractions on the boundary $\alpha=\alpha_1$ are in equilibrium, we must have the relations,

$$\left.\begin{aligned} A_1 \sinh \alpha_1 - D_1 \cosh \alpha_1 &= 0 \\ B_1 \cosh \alpha_1 + C_1 \sinh \alpha_1 &= 0 \\ B_2 \quad + \quad 2\, C_0 \sinh 2\alpha_1 &= 0 \end{aligned}\right\} \tag{27}$$

Since the plate extends to infinity and the stress there is zero, we take

$$\chi = \phi_0 + \sum_{n=1}^{\infty} \phi_n \cos n\beta + \sum_{n=1}^{\infty} \psi_n \sin n\beta \tag{28}$$

where

$$\phi_0 = b_0 e^{-2\alpha} + c_0 \alpha,$$
$$\phi_1 = b_1 e^{-3\alpha} + b'_1 \sinh \alpha,$$
$$\phi_2 = b_0 + b_2 e^{-4\alpha} + b'_2 e^{-2\alpha}, \tag{29}$$

and for $n \geqq 3$,

$$\phi_n = b_{n-2}\, e^{-(n-2)\alpha} + b_n\, e^{-(n+2)\alpha} + b'_n\, e^{-n\alpha},$$

and

$$\psi_1 = d_1\, e^{-3\alpha} + d'_1 \cosh \alpha,$$
$$\psi_2 = d_0 + d_2\, e^{-4\alpha} + d'_2\, e^{-2\alpha}, \tag{30}$$

and for $n \geq 3$,

$$\psi_n = d_{n-2}\, e^{-(n-2)\alpha} + d_n\, e^{-(n+2)\alpha} + d'_n\, e^{-n\alpha}.$$

To simplify the problem, we consider the terms in the surface tractions (26) separately.

(*i*) $\widehat{\alpha\alpha} = A_0, \quad \widehat{\alpha\beta} = 0, \quad$ when $\alpha=\alpha_1$.

In this case, we omit the ψ's and the odd ϕ's from the expression for χ. Then we have from (24) and (25), when $\alpha=\alpha_1$,

$$\phi'_0 \sinh 2\alpha + 3\phi_2 = \frac{c^2}{4} A_0 (1 + 2 \cosh^2 2\alpha),$$

$$-4\phi_2 \cosh 2\alpha + \phi'_2 \sinh 2\alpha + 10\phi_4 = -c^2 A_0 \cosh 2\alpha.$$

$$-16\phi_4 \cosh 2\alpha + \phi'_4 \sinh 2\alpha + 21\phi_6 + \phi_2 = \frac{c^2}{4} A_0,$$

and for $n \geqq 3$,

$$-4n^2\phi_{2n}\cosh 2a+\phi'_{2n}\sinh 2a+\tfrac{1}{2}(2n+2)(2n+3)\phi_{2n+2}$$
$$+\tfrac{1}{2}(2n-2)(2n-3)\phi_{2n-2}=0,$$

and

$$\phi'_0+2\phi'_2\cosh 2a-2\phi_2\sinh 2a-\tfrac{5}{2}\phi'_4=0,$$

and for $n \geqq 2$,

$$-\tfrac{1}{2}(2n-3)\phi'_{2n-2}+2n\phi'_{2n}\cosh 2a-2n\,\phi_{2n}\sinh 2a$$
$$-\tfrac{1}{2}(2n+3)\,\phi'_{2n+2}=0.$$

These equations are all satisfied by

$$\phi'_0=\frac{c^2}{2}A_0\sinh 2a,\qquad \phi_2=\frac{c^2}{4}A_0,\quad \phi'_2=0,$$

and for $n \geqq 2$, $\qquad \phi_{2n}=0,\quad \phi'_{2n}=0,$

a being replaced by a_1 in these expressions.

Hence

$$-2b_0e^{-2a}+c_0=\frac{c^2}{2}A_0\sinh 2a,$$
$$b_0+b_2e^{-4a}+b'_2e^{-2a}=\frac{c^2}{4}A_0,$$
$$-4b_2e^{-4a}-2b'_2e^{-2a}=0,$$

and for $n \geqq 2$,

$$b_{2n-2}e^{-(2n-2)a}+b_{2n}e^{-(2n+2)a}+b'_{2n}e^{-2na}=0,$$
$$(n-1)\,b_{2n-2}\,e^{-(2n-2)a}+(n+1)\,b_{2n}\,e^{-(2n+2)a}+n\,b'_{2n}\,e^{-2na}=0.$$

From the last two equations, we get

$$b_{2n}=b_{2n-2}\,e^{4a},\quad b'_{2n}=-2b_{2n-2}\,e^{2a},$$

so that $\qquad b_{2n}=b_2\,e^{(4n-4)a},\quad b'_{2n}=-2b_2\,e^{(4n-6)a},$

where we replace a by a_1.

Now $\quad b_{2n}\,e^{-2na_1}\to 0,\ b'_{2n}\,e^{-2na_1}\to 0$ as $n\to\infty$, therefore $b_2=0$ and hence $b_{2n}=0$, $b'_{2n}=0$ for $n \geqq 2$. Substituting $b_2=0$ in the first three equations, we get

$$b'_2=0,\quad b_0=\frac{c^2}{4}A_0,\qquad c_0=\frac{c^2}{2}A_0\cosh 2a_1.$$

(*ii*) $\widehat{\alpha\alpha}=A_2\cos 2\beta, \quad \widehat{\alpha\beta}=0,$ when $\alpha=\alpha_1$.

Here also we omit the ψ's and the odd ϕ's.

We have when $\alpha=\alpha_1$,

$$\phi'_0 \sinh 2\alpha + 3\phi_2 = -\frac{c^2}{2}A_2\cosh 2\alpha,$$

$$-4\phi_2\cosh 2\alpha + \phi'_2\sinh 2\alpha + 10\phi_4 = \frac{c^2}{4}A_2\left(\tfrac{3}{2}+2\cosh^2 2\alpha\right),$$

$$-16\phi_4\cosh 2\alpha + \phi'_4\sinh 2\alpha + 21\phi_6 + \phi_2 = -\frac{c^2}{2}A_2\cosh 2\alpha,$$

$$-36\phi_6\cosh 2\alpha + \phi'_6\sinh 2\alpha + 36\phi_8 + 6\phi_4 = \frac{c^2}{8}A_2,$$

and for $n \geqq 4$,

$$-4n^2\phi_{2n}\cosh 2\alpha + \phi'_{2n}\sinh 2\alpha + \tfrac{1}{2}(2n+2)(2n+3)\phi_{2n+2} + \tfrac{1}{2}(2n-2)(2n-3)\phi_{2n-2} = 0,$$

and

$$\phi'_0 + 2\phi'_2\cosh 2\alpha - 2\phi_2\sinh 2\alpha - \tfrac{5}{2}\phi'_4 = 0,$$

and for $n \geqq 2$,

$$-\tfrac{1}{2}(2n-3)\phi'_{2n-2} + 2n\,\phi'_{2n}\cosh 2\alpha - 2n\phi_{2n}\sinh 2\alpha - \tfrac{1}{2}(2n+3)\phi'_{2n+2} = 0.$$

These equations are all satisfied by

$$\phi'_0 = 0,$$

$$\phi_2 = -\frac{c^2}{6}A_2\cosh 2\alpha, \qquad \phi'_2 = -\frac{c^2}{6}A_2\sinh 2\alpha,$$

$$\phi_4 = \frac{c^2}{48}A_2, \qquad \phi'_4 = 0,$$

and for $n \geqq 3$, $\qquad \phi_{2n}=0, \qquad \phi'_{2n}=0,$

α being replaced by α_1.

Proceeding as in case (*i*), we get

$$b_{2n} = b_4 e^{(4n-8)\alpha}, \qquad b'_{2n} = -2b_4 e^{(4n-10)\alpha},$$

α being replaced by α_1.

Now as $b_{2n}\, e^{-2n\alpha_1} \longrightarrow 0$, $b'_{2n}\, e^{-2n\alpha_1} \longrightarrow 0$, as $n \longrightarrow \infty$, therefore $b_4=0$ and hence $b_{2n}=0$, $b'_{2n}=0$ for $n \geqq 3$.

Then from the other equations, we easily find

$$b_0 = -\frac{c^2}{8} A_2 e^{2\alpha}, \qquad c_0 = -\frac{c^2}{4} A_2,$$

$$b_2 = \frac{c^2}{24} A_2 e^{2\alpha}, \qquad b'_2 = \frac{c^2}{24} A_2 (e^{4\alpha} - 3),$$

$$b'_4 = -\frac{c^2}{48} A_2 e^{4\alpha},$$

α being replaced by α_1.

(iii) $\widehat{\alpha\alpha} = A_4 \cos 4\beta, \quad \widehat{\alpha\beta} = 0,$ when $\alpha = \alpha_1$.

We omit as before the ψ's and the odd ϕ's.

Proceeding exactly as in cases (i) and (ii), we have when $\alpha = \alpha_1$,

$$\phi'_0 = 0, \quad \phi_2 = \frac{c^2}{24} A_4, \quad \phi'_2 = 0,$$

$$\phi_4 = -\frac{c^2}{30} A_4 \cosh 2\alpha, \quad \phi'_4 = -\frac{c^2}{30} A_4 \sinh 2\alpha,$$

$$\phi_6 = \frac{c^2}{120} A_4, \quad \phi'_6 = 0,$$

and for $n \geqq 4$, $\qquad \phi_{2n} = 0, \qquad \phi'_{2n} = 0.$

We can then show as before

$$b_{2n} = b_6 e^{(4n-12)\alpha}, \qquad b'_{2n} = -2b_6 e^{(4n-14)\alpha},$$

α being replaced by α_1.

Now as $b_{2n}\, e^{-2n\alpha_1} \longrightarrow 0$, $b'_{2n}\, e^{-2n\alpha_1} \longrightarrow 0$ when $n \longrightarrow \infty$, we must have $b_6 = 0$ and therefore $b_{2n} = 0$, $b'_{2n} = 0$ for $n \geqq 4$.

Substituting $b_6 = 0$ in the other equations, we get

$$b_0 = 0, \; c_0 = 0, \; b_2 = -\frac{c^2}{24} A_4 e^{4\alpha}, \; b'_2 = \frac{c^2}{12} A_4 e^{2\alpha},$$

$$b_4 = \frac{c^2}{40} A_4 e^{4\alpha}, \; b'_4 = \frac{c^2}{120} A_4 (3e^{6\alpha} - 5e^{2\alpha}), \qquad b'_6 = -\frac{c^2}{60} A_4 e^{6\alpha}$$

α being replaced by α_1.

(iv) $\widehat{\alpha\alpha} = A_{2m} \cos 2m\beta, \quad \widehat{\alpha\beta} = 0, \quad$ when $\alpha = \alpha_1$.

We omit as before the ψ's and the odd ϕ's.

Proceeding exactly as in the previous cases, we have when $\alpha=\alpha_1$,

$$\phi'_0=0, \quad \phi_{2m-2}=\frac{c^2 A_{2m}}{4(2m-1)(2m-2)}, \quad \phi'_{2m-2}=0,$$

$$\phi_{2m}=-\frac{c^2 A_{2m}}{2(4m^2-1)}\cosh 2\alpha, \quad \phi'_{2m}=-\frac{c^2 A_{2m}}{2(4m^2-1)}\sinh 2\alpha$$

$$\phi_{2m+2}=\frac{c^2 A_{2m}}{4(2m+1)(2m+2)}, \qquad \phi'_{2m+2}=0,$$

and for $n \geqq m+2$ and $2 \leqq n \leqq m-2$,

$$\phi_{2n}=0, \quad \phi'_{2n}=0.$$

We can show as before

$$b_{2n}=b_{2m+2}\, e^{(4n-4m-4)\alpha}, \qquad b'_{2n}=-2b_{2m+2}\, e^{(4n-4m-6)\alpha}.$$

Now as $b_{2n}e^{-2n\alpha_1}\longrightarrow 0$, $b'_{2n}e^{-2n\alpha_1}\longrightarrow 0$ when $n\longrightarrow\infty$, we must have $b_{2m+2}=0$, so that $b_{2n}=0$, $b'_{2n}=0$ for $n\geqq m+2$.

Substituting $b_{2m+2}=0$ in the equations with ϕ whose suffixes lie between $2m-4$ and $2m+4$, we have,

$$b_{2m-4}=0, \quad b_{2m-2}=-\frac{c^2 A_{2m}}{8(2m-1)}\, e^{2m\alpha},$$

$$b'_{2m-2}=\frac{mc^2 A_{2m}}{4(2m-1)(2m-2)}\, e^{(2m-2)\alpha},$$

$$b_{2m}=\frac{c^2 A_{2m}}{8(2m+1)}\, e^{2m\alpha},$$

$$b'_{2m}=\frac{c^2 A_{2m}}{8(4m^2-1)}\left[(2m-1)\, e^{2\alpha}-(2m+1)\, e^{-2\alpha}\right] e^{2m\alpha},$$

$$b'_{2m+2}=-\frac{mc^2 A_{2m}}{4(2m+1)(2m+2)}\, e^{(2m+2)\alpha},$$

α being replaced by α_1.

Again, substituting $b_{2m-4}=0$ in the equations with ϕ whose suffixes are $\leqq 2m-4$, and solving, we get

$$b_0=0,\ c_0=0 \quad\text{and}\quad b_{2n}=0,\ b'_{2n}=0 \quad\text{for}\quad n\leqq m-2.$$

(v) $\widehat{\alpha\alpha}=A_1\cos\beta$, $\widehat{\alpha\beta}=D_1\sin\beta$, when $\alpha=\alpha_1$.

Here we omit the ψ's and the even ϕ's.

Since the tractions on $\alpha=\alpha_1$, form a system in equilibrium, we have

$$A_1\sinh\alpha_1-D_1\cosh\alpha_1=0,$$

so that we can write $A_1 = k_1 \cosh \alpha_1$, $D_1 = k_1 \sinh \alpha_1$.

Hence we have, when $\alpha = \alpha_1$,

$$-\phi_1 \cosh 2\alpha + \phi'_1 \sinh 2\alpha + \phi_1 + 6\phi_3$$

$$= \frac{c^2}{4} k_1 \cosh \alpha \, (1 + 2\cosh^2 2\alpha - 2\cosh 2\alpha),$$

$$-9\phi_3 \cosh 2\alpha + \phi'_3 \sinh 2\alpha + 15\phi_5 = \frac{c^2}{4} k_1 \cosh \alpha \, (-2\cosh 2\alpha + \tfrac{1}{2}),$$

$$-25\phi_5 \cosh 2\alpha + \phi'_5 \sinh 2\alpha + 28\phi_7 + 3\phi_3 = \frac{c^2}{8} k_1 \cosh \alpha,$$

and for $n \geqq 3$,

$$-(2n+1)^2 \phi_{2n+1} \cosh 2\alpha + \phi'_{2n+1} \sinh 2\alpha + \tfrac{1}{2}(2n+3)(2n+4)\phi_{2n+3}$$
$$+ \tfrac{1}{2}(2n-1)(2n-2)\phi_{2n-1} = 0$$

and

$$\phi'_1 \cosh 2\alpha - \phi_1 \sinh 2\alpha + \phi'_1 - 2\phi'_3$$

$$= \frac{c^2}{4} k_1 \sinh \alpha (1 + 2\cosh^2 2\alpha + 2\cosh 2\alpha),$$

$$3\phi'_3 \cosh 2\alpha - 3\phi_3 \sinh 2\alpha - 3\phi'_5 = \frac{c^2}{4} k_1 \sinh \alpha \, (-2\cosh 2\alpha - \tfrac{1}{2})$$

$$-\phi'_3 + 5\phi'_5 \cosh 2\alpha - 5\phi_5 \sinh 2\alpha - 4\phi'_7 = \frac{c^2}{8} k_1 \sinh \alpha,$$

and for $n \geqq 3$,

$$-\tfrac{1}{2}(2n-2)\phi'_{2n-1} + (2n+1)\phi'_{2n+1} \cosh 2\alpha - (2n+1)\phi_{2n+1} \sinh 2\alpha$$
$$- \tfrac{1}{2}(2n+4)\phi'_{2n+3} = 0.$$

These equations are all salisfied by

$$\phi'_1 \cosh \alpha - \phi_1 \sinh \alpha = \frac{c^2}{8} k_1 \sinh 4\alpha,$$

$$\phi_3 = \frac{c^2}{24} k_1 \cosh \alpha, \quad \phi'_3 = -\frac{c^2}{8} k_1 \sinh \alpha,$$

and for $n \geqq 2$, $\phi_{2n+1} = 0$, $\phi'_{2n+1} = 0$, α being everywhere replaced by α_1.

Proceeding as in the previous cases, we have

$$b_{2n+1} = b_3 e^{(4n-4)\alpha}, \quad b'_{2n+1} = -2b_3 e^{(4n-6)\alpha}.$$

Now as $b_{2n+1}e^{-(2n+1)\alpha_1} \longrightarrow 0$, $b'_{2n+1}e^{-(2n+1)\alpha_1} \longrightarrow 0$, when $n \longrightarrow \infty$, we must have $b_3=0$, so that

$$b_{2n+1}=0, \quad b'_{2n+1}=0 \quad \text{for } n \geqq 2.$$

The other equations then give

$$b_1 = \frac{k_1 c^2}{16}, \qquad b'_1 = \frac{k_1 c^2}{16}(2e^{-2\alpha}+e^{4\alpha}),$$

$$b'_3 = \frac{k_1 c^2}{48}(e^{4\alpha}-2e^{2\alpha}).$$

where we replace α by α_1.

(*vi*) $\widehat{\alpha\alpha} = A_3 \cos 3\beta$, $\widehat{\alpha\beta}=0$, when $\alpha=\alpha_1$.

We omit the ψ's and the even ϕ's.

Proceeding exactly as in case (v), we have, when $\alpha=\alpha_1$,

$$\phi'_1 \cosh\alpha - \phi_1 \sinh\alpha = -\frac{c^2}{8} A_3 \sinh\alpha,$$

$$\phi_3 = -\frac{c^2}{16} A_3 \cosh 2\alpha, \qquad \phi'_3 = -\frac{c^2}{16} A_3 \sinh 2\alpha,$$

$$\phi_5 = \frac{c^2}{80} A_3, \qquad \phi'_5 = 0,$$

and for $n \geqq 3$, $\phi_{2n+1}=0$, $\phi'_{2n+1}=0$.

As before, we find

$$b_1 = -\frac{c^2}{16} A_3 e^{3\alpha}, \qquad b'_1 = -\frac{3c^2}{16} A_3 e^{\alpha},$$

$$b_3 = \frac{c^2}{32} A_3 e^{3\alpha}, \qquad b'_3 = \frac{c^2}{32} A_3 (e^{5\alpha}-2e^{\alpha}),$$

$$b_5=0, \qquad b'_5 = -\frac{3c^2}{160} A_3 e^{5\alpha},$$

and for $n \geqq 3$, $b_{2n+1}=0$, $b'_{2n+1}=0$,

α being replaced by α_1.

(*vii*) $\widehat{\alpha\alpha} = A_{2m+1} \cos(2m+1)\beta$, $\widehat{\alpha\beta}=0$, when $\alpha=\alpha_1$.

We omit the ψ's and the even ϕ's.

We have

$$\phi'_1 \cosh\alpha - \phi_1 \sinh\alpha = 0,$$

$$\phi_{2m-1} = \frac{c^2 A_{2m+1}}{8m(2m-1)}, \quad \phi'_{2m-1} = 0,$$

$$\phi_{2m+1} = -\frac{c^2 A_{2m+1}}{4m(2m+2)} \cosh 2\alpha,$$

$$\phi'_{2m+1} = -\frac{c^2 A_{2m+1}}{4m(2m+2)} \sinh 2\alpha,$$

$$\phi_{2m+3} = \frac{c^2 A_{2m+1}}{4(2m+2)(2m+3)}, \quad \phi'_{2m+3} = 0,$$

and $\phi_{2n+1} = 0$, $\phi'_{2n+1} = 0$ for $n \geqq m+2$ and $n \leqq m-2$, α being replaced by α_1.

Solving these equations, we have

$$b_{2m-1} = -\frac{c^2 A_{2m+1}}{16m} e^{(2m+1)\alpha},$$

$$b'_{2m-1} = \frac{(2m+1)c^2}{16m(2m-1)} A_{2m+1} e^{(2m-1)\alpha},$$

$$b_{2m+1} = \frac{c^2 A_{2m+1}}{16(m+1)} e^{(2m+1)\alpha},$$

$$b'_{2m+1} = \frac{c^2 A_{2m+1}}{16m(m+1)} [m e^{2\alpha} - (m+1)e^{-2\alpha}] e^{(2m+1)\alpha},$$

$$b_{2m+3} = 0, \quad b'_{2m+3} = -\frac{(2m+1)c^2}{8(2m+2)(2m+3)} A_{2m+1} e^{(2m+3)\alpha},$$

and $b_{2n+1} = 0$, $b'_{2n+1} = 0$, for $n \geqq m+2$ and $n \leqq m-2$, α being replaced by α_1.

(*viii*) $\widehat{\alpha\alpha} = 0$, $\widehat{\alpha\beta} = D_2 \sin 2\beta$, when $\alpha = \alpha_1$.

We omit the ψ's and the odd ϕ's.

We have, when $\alpha = \alpha_1$,

$$\phi'_0 = -\frac{c^2}{4} D_2,$$

$$\phi_2 = \frac{c^2}{12} D_2 \sinh 2\alpha, \quad \phi'_2 = \frac{c^2}{3} D_2 \cosh 2\alpha,$$

$$\phi_4 = 0, \quad \phi'_4 = -\frac{c^2}{12} D_2,$$

and $\phi_{2n} = 0$, $\phi'_{2n} = 0$ for $n \geqq 3$.

Solving these equations, as before, we find

$$b_0 = \frac{c^2}{8} D_2 e^{2\alpha}, \quad c_0 = 0,$$

$$b_2 = -\frac{c^2}{24} D_2 e^{2\alpha}, \qquad b'_2 = -\frac{c^2}{12} D_2 e^{4\alpha},$$

$$b_4 = 0, \quad b'_4 = \frac{c^2}{24} D_2 e^{4\alpha},$$

and for $n \geqq 3$, $b_{2n} = 0$, $b'_{2n} = 0$, α being replaced by α_1.

(*ix*) $\widehat{\alpha\alpha} = 0$, $\widehat{\alpha\beta} = D_{2m} \sin 2m\beta$, when $\alpha = \alpha_1$.

We omit the ψ's and the odd ϕ's.

We have then, for $\alpha = \alpha_1$,

$$\phi'_0 = 0, \quad \phi_{2m-2} = 0, \quad \phi'_{2m-2} = -\frac{c^2 D_{2m}}{4(2m-1)},$$

$$\phi_{2m} = \frac{c^2 D_{2m}}{4m(4m^2-1)} \sinh 2\alpha,$$

$$\phi'_{2m} = \frac{mc^2 D_{2m}}{4m^2-1} \cosh 2\alpha, \quad \phi_{2m+2} = 0, \quad \phi'_{2m+2} = -\frac{c^2 D_{2m}}{4(2m+1)},$$

and $\phi_{2n} = 0$, $\phi'_{2n} = 0$ for $n \geqq m+2$ and $2 \leqq n \leqq m-2$.

Solving these equations as before, we find

$$b_0 = 0, \quad c_0 = 0, \quad b_{2m-2} = \frac{c^2 D_{2m}}{8(2m-1)} e^{2m\alpha},$$

$$b'_{2m-2} = -\frac{c^2 D_{2m}}{8(2m-1)} e^{(2m-2)\alpha}, \qquad b_{2m} = -\frac{c^2 D_{2m}}{8(2m+1)} e^{2m\alpha},$$

$$b'_{2m} = -\frac{c^2 D_{2m}}{8m(4m^2-1)} [(2m-1)(m+1) e^{2\alpha}$$

$$-(2m+1)(m-1) e^{-2\alpha}] e^{2m\alpha},$$

$$b_{2m+2} = 0, \quad b'_{2m+2} = \frac{c^2 D_{2m}}{8(2m+1)} e^{(2m+2)\alpha},$$

and $b_{2n} = 0$, $b'_{2n} = 0$ for $n \geqq m+2$ and $n \leqq m-2$, α being replaced by α_1.

(*x*) $\widehat{\alpha\alpha} = 0$, $\widehat{\alpha\beta} = D_3 \sin 3\beta$, when $\alpha = \alpha_1$.

We omit the ψ's and the even ϕ's.

We have then, as before,

$$\phi'_1 \cosh \alpha - \phi_1 \sinh \alpha = -\frac{c^2}{8} D_3 \cosh \alpha ,$$

$$\phi_3 = \frac{c^2}{48} D_3 \sinh 2\alpha, \quad \phi'_3 = \frac{3c^2}{16} D_3 \cosh 2\alpha,$$

$$\phi_5 = 0, \quad \phi'_5 = -\frac{c^2}{16} D_3 ,$$

and for $n \geqq 3$, $\phi_{2n+1} = 0$, $\phi'_{2n+1} = 0$, α being replaced by α_1.

These equations give

$$b_1 = \frac{c^2}{16} D_3 e^{3\alpha}, \quad b'_1 = \frac{c^2}{16} D_3 e^{\alpha},$$

$$b_3 = -\frac{c^2}{32} D_3 e^{3\alpha}, \quad b'_3 = -\frac{c^2}{96} D_3 (5 e^{2\alpha} - 2e^{-2\alpha}) e^{3\alpha},$$

$$b_5 = 0, \quad b'_5 = \frac{c^2}{32} D_3 e^{5\alpha},$$

and for $n \geqq 3$, $b_{2n+1} = 0$, $b'_{2n+1} = 0$, α being replaced by α_1.

(xi) $\widehat{\alpha\alpha} = 0$, $\widehat{\alpha\beta} = D_{2m+1} \sin (2m+1)\beta$, when $\alpha = \alpha_1$.

We omit the ψ's and the even ϕ's.

We have then for $\alpha = \alpha_1$,

$$\phi'_1 \cosh \alpha - \phi_1 \sinh \alpha = 0,$$

$$\phi_{2m-1} = 0, \quad \phi'_{2m-1} = -\frac{c^2 D_{2m+1}}{8m},$$

$$\phi_{2m+1} = \frac{c^2 D_{2m+1}}{4m(2m+1)(2m+2)} \sinh 2\alpha,$$

$$\phi'_{2m+1} = \frac{(2m+1)c^2 D_{2m+1}}{4m(2m+2)} \cosh 2\alpha,$$

$$\phi_{2m+3} = 0, \quad \phi'_{2m+3} = -\frac{c^2 D_{2m+1}}{8(m+1)},$$

and $\phi_{2n+1} = 0$, $\phi'_{2n+1} = 0$ for $n \geqq m+2$ and $1 \leqq n \leqq m-2$.

These equations give

$$b_{2m-1}=\frac{c^2D_{2m+1}}{16m}e^{(2m+1)\alpha},\quad b'_{2m-1}=-\frac{c^2D_{2m+1}}{16m}e^{(2m-1)\alpha},$$

$$b_{2m+1}=-\frac{c^2D_{2m+1}}{16(m+1)}e^{(2m+1)\alpha},$$

$$b'_{2m+1}=-\frac{c^2D_{2m+1}}{16m(2m+1)(2m+2)}\left[2m(2m+3)e^{2\alpha}-(2m-1)(2m+2)e^{-2\alpha}\right]e^{(2m+1)\alpha},$$

$$b_{2m+3}=0,\quad b'_{2m+3}=\frac{c^2D_{m+1}}{16(m+1)}e^{(2m+3)\alpha},$$

and $\quad b_{2n+1}=0,\ b'_{2n+1}=0$ for $n\geqq m+2$ and $0\leqq n\leqq m-2$, α being replaced by α_1.

(*xii*) $\widehat{\alpha\alpha}=B_2\sin 2\beta,\ \widehat{\alpha\beta}=C_0$, when $\alpha=\alpha_1$.

Here we omit the ϕ's and the odd ψ's.

Since the tractions on $\alpha=\alpha_1$, form a system in equilibrium, we have

$$B_2+2C_0\sinh 2\alpha_1=0.$$

We have then, for $\alpha=\alpha_1$,

$$\psi_2=-\frac{c^2}{6}B_2\cosh 2\alpha,\quad \psi'_2=\frac{c^2}{2}C_0-\frac{c^2}{6}B_2\sinh 2\alpha,$$

$$\psi_4=\frac{c^2}{48}B_2,\quad \psi'_4=0,$$

and for $n\geqq 3,\quad \psi_{2n}=0,\ \psi'_{2n}=0.$

Solving these equations as before, we find

$$d_0=\frac{c^2}{4}.C_0-\frac{c^2}{8}B_2e^{2\alpha},\quad d_2=\frac{c^2}{24}B_2e^{2\alpha},$$

$$d'_2=\frac{c^2}{24}B_2(e^{4\alpha}-3)-\frac{c^2}{4}C_0e^{2\alpha},$$

$$d_4=0,\quad d'_4=-\frac{c^2}{48}B_2e^{4\alpha},$$

and for $n\geqq 3,\qquad d_{2n}=0.\ d'_{2n}=0,$
α being replaced by α_1.

(*xiii*) $\widehat{\alpha\alpha}=B_4\sin 4\beta,\ \widehat{\alpha\beta}=0$, when $\alpha=\alpha_1$.

We omit the ϕ's and the odd ψ's.

We have then, for $\alpha=\alpha_1$,

$$\psi_2=\frac{c^2}{24}B_4,\quad \psi'_2=0,$$

$$\psi_4=-\frac{c^2}{30}B_4\cosh 2\alpha,\quad \psi'_4=-\frac{c^2}{30}B_4\sinh 2\alpha,$$

$$\psi_6=\frac{c^2}{120}B_4,\quad \psi'_6=0,$$

and for $n\geqq 4,\quad \psi_{2n}=0,\quad \psi'_{2n}=0.$

Solving these equations as before, we find

$$d_0=0,\quad d_2=-\frac{c^2}{24}B_4e^{4\alpha},\quad d'_2=\frac{c^2}{12}B_4e^{2\alpha},$$

$$d_4=\frac{c^2}{40}B_4\,e^{4\alpha},\quad d'_4=\frac{c^2}{120}B_4(3e^{2\alpha}-5e^{-2\alpha})\,e^{4\alpha},$$

$$d_6=0,\quad d'_6=-\frac{c^2}{60}B_4\,e^{6\alpha},$$

and for $n\geqq 4,\quad d_{2n}=0,\ d'_{2n}=0,$
α being replaced by α_1.

(*xiv*) $\widehat{\alpha\alpha}=B_{2m}\sin 2m\beta,\quad \widehat{\alpha\beta}=0,\quad$ when $\alpha=\alpha_1$.
We omit the ϕ's and the odd ψ's.

We have then, for $\alpha=\alpha_1$,

$$\psi_{2m-2}=\frac{c^2B_{2m}}{4(2m-1)(2m-2)},\quad \psi'_{2m-2}=0,$$

$$\psi_{2m}=-\frac{c^2B_{2m}}{2(4m^2-1)}\cosh 2\alpha,\quad \psi'_{2m}=-\frac{c^2B_{2m}}{2(4m^2-1)}\sinh 2\alpha,$$

$$\psi_{2m+2}=\frac{c^2B_{2m}}{4(2m+1)(2m+2)},\quad \psi'_{2m+2}=0,$$

and $\quad \psi_{2n}=0,\ \psi'_{2n}=0,$ for $n\geqq m+2$ and $n\leqq m-2$.

Solving these equations, we find

$$d_{2m-2}=-\frac{c^2B_{2m}}{8(2m-1)}e^{2m\alpha},\quad d'_{2m-2}=\frac{mc^2B_{2m}}{4(2m-1)(2m-2)}e^{(2m-2)\alpha},$$

$$d_{2m}=\frac{c^2B_{2m}}{8(2m+1)}e^{2m\alpha},$$

$$d'_{2m}=\frac{c^2B_{2m}}{8(4m^2-1)}\left[(2m-1)e^{112}-(2m+1)e^{-2\alpha}\right]e^{2m\alpha},$$

$$d_{2m+2}=0,\quad d'_{2m+2}=-\frac{mc^2B_{2m}}{4(2m+1)(2m+2)}e^{(2m+2)\alpha},$$

and $d_{2n}=0,\ d'_{2n}=0$ for $n \geqq m+2$ and $n \leqq m-2$, α being replaced by α_1.

(*xv*) $\widehat{\alpha\alpha}=B_1 \sin\beta,\quad \widehat{\alpha\beta}=C_1\cos\beta,$ when $\alpha=\alpha_1$.
We omit the ϕ's and the even ψ's.

Since the tractions on $\alpha=\alpha_1$, form a system in equilibrium, we have

$$B_1 \cosh\alpha_1 + C_1 \sinh\alpha_1 = 0,$$

so that $$B_1 = -k_2 \sinh\alpha_1,\quad C_1 = k_2\cosh\alpha_1.$$

We have then, for $\alpha=\alpha_1$,

$$\psi_1\cosh\alpha - \psi'_1\sinh\alpha = \frac{c^2}{8}k_2\sinh 4\alpha,$$

$$\psi_3 = -\frac{c^2}{24}k_2\sinh\alpha,\qquad \psi'_3=\frac{c^2}{8}k_2\cosh\alpha,$$

and for $n \geqq 2$, $\quad \psi_{2n+1}=0,\ \psi'_{2n+1}=0.$

Solving these equations, we get

$$d_1=\frac{k_2c^2}{16},\qquad d'_1=\frac{k_2c^2}{16}(e^{4\alpha}-2e^{-2\alpha}),$$

$$d_3=0,\quad d'_3=-\frac{k_2c^2}{48}e^{3\alpha}(e^{\alpha}+2e^{-\alpha}),$$

and for $n \geqq 2$, $\quad d_{2n+1}=0,\ d'_{2n+1}=0$, α being replaced by α_1.

(*xvi*) $\widehat{\alpha\alpha}=B_3\sin 3\beta,\quad \widehat{\alpha\beta}=0,$ when $\alpha=\alpha_1$.
We omit the ϕ's and the even ψ's.

We have then, for $\alpha=\alpha_1$,

$$\psi'_1\sinh\alpha-\psi_1\cosh\alpha = -\frac{c^2}{8}B_3\cosh\alpha,$$

$$\psi_3=-\frac{c^2}{16}B_3\cosh 2\alpha,\quad \psi'_3=-\frac{c^2}{16}B_3\sinh 2\alpha,$$

$$\psi_5=\frac{c^2}{80}B_3,\qquad \psi'_5=0,$$

and for $n \geqq 3$, $\quad \psi_{2n+1}=0,\ \psi'_{2n+1}=0.$

Solving these equations, we get

$$d_1=-\frac{c^2}{16}B_3e^{3\alpha},\qquad d'_1=\frac{3c^2}{16}B_3e^{\alpha},$$

$$d_3=\frac{c^2}{32}B_3e^{3\alpha},\quad d'_3=\frac{c^2}{32}B_3(e^{5\alpha}-2e^{\alpha}),$$

$$d_5=0,\quad d'_5=-\frac{3c^2}{160}B_3e^{5\alpha},$$

and for $n \geqq 3$, $\quad d_{2n+1}=0,\ d'_{2n+1}=0,\quad$ α being replaced by α_1.

(*xvii*) $\widehat{\alpha\alpha}=B_{2m+1}\sin(2m+1)\beta$, $\widehat{\alpha\beta}=0$, when $\alpha=\alpha_1$

We omit the ϕ's and the even ψ's.

We have then, for $\alpha=\alpha_1$,

$$\psi'_1\sinh\alpha-\psi_1\cosh\alpha=0$$

$$\psi_{2m-1}=\frac{c^2 B_{2m+1}}{8m(2m-1)},\quad \psi'_{2m-1}=0,$$

$$\psi_{2m+1}=-\frac{c^2 B_{2m+1}}{8m(m+1)}\cosh 2\alpha,$$

$$\psi'_{2m+1}=-\frac{c^2 B_{2m+1}}{8m(m+1)}\sinh 2\alpha,$$

$$\psi_{2m+3}=\frac{c^2 B_{2m+1}}{4(2m+2)(2m+3)},\quad \psi'_{2m+3}=0,$$

and $\psi_{2n+1}=0$, $\psi'_{2n+1}=0$ for $n\geqq m+2$ and $1\leqq n\leqq m-2$.

Solving these equations, we have

$$d_{2m-1}=-\frac{c^2 B_{2m+1}}{16m}e^{(2m+1)\alpha},$$

$$d'_{2m-1}=\frac{(2m+1)c^2 B_{2m+1}}{16m(2m-1)}e^{(2m-1)\alpha},$$

$$d_{2m+1}=\frac{c^2 B_{2m+1}}{16(m+1)}e^{(2m+1)\alpha},$$

$$d'_{2m+1}=\frac{c^2 B_{2m+1}}{16m(m+1)}\left[m e^{2\alpha}-(m+1)e^{-2\alpha}\right]e^{(2m+1)\alpha},$$

$$d_{2m+3}=0,\quad d'_{2m+3}=-\frac{(2m+1)c^2 B_{2m+1}}{8(2m+2)(2m+3)}e^{(2m+1)\alpha},$$

and $d_{2n+1}=0$, $d'_{2n+1}=0$ for $n\geqq m+2$ and $n\leqq m-2$, α being replaced by α_1.

(*xviii*) $\widehat{\alpha\alpha}=0$, $\widehat{\alpha\beta}=C_2\cos 2\beta$, when $\alpha=\alpha_1$.

We omit the ϕ's and the odd ψ's.

We have then, for $\alpha=\alpha_1$,

$$\psi_2=-\frac{c^2}{12}C_2\sinh 2\alpha,\quad \psi'_2=-\frac{c^2}{3}C_2\cosh 2\alpha,$$

$$\psi_4=0,\quad \psi'_4=\frac{c^2}{12}C_2,$$

and for $n\geqq 3$, $\psi_{2n}=0$, $\psi'_{2n}=0$.

Solving these equations, we find

$$d_0 = -\frac{c^2}{8} C_2 e^{2\alpha},$$

$$d_2 = \frac{c^2}{24} C_2 e^{2\alpha}, \quad d'_2 = \frac{c^2}{12} C_2 e^{4\alpha},$$

$$d_4 = 0, \quad d'_4 = -\frac{c^2}{24} C_2 e^{4\alpha},$$

and for $n \geqq 3$, $\quad d_{2n} = 0, \quad d'_{2n} = 0,$ α being replaced by α_1.

(*xix*) $\quad \widehat{\alpha\alpha} = 0, \quad \widehat{\alpha\beta} = C_4 \cos 4\beta,$ when $\alpha = \alpha_1$.

We omit the ϕ's and the odd ψ's.

We have then, for $\alpha = \alpha_1$,

$$\psi_2 = 0, \quad \psi'_2 = \frac{c^2}{12} C_4,$$

$$\psi_4 = -\frac{c^2}{120} C_4 \sinh 2\alpha, \qquad \psi'_4 = -\frac{2c^2}{15} C_4 \cosh 2\alpha,$$

$$\psi_6 = 0, \quad \psi'_6 = \frac{c^2}{20} C_4,$$

and for $n \geqq 4$, $\psi_{2n} = 0, \; \psi'_{2n} = 0.$

Solving these equations, we find

$$d_0 = 0, \quad d_2 = -\frac{c^2}{24} C_4 e^{4\alpha},$$

$$d'_2 = \frac{c^2}{24} C_4 e^{2\alpha}, \quad d_4 = \frac{c^2}{40} C_4 e^{4\alpha},$$

$$d'_4 = \frac{c^2}{240} C_4 (9e^{6\alpha} - 5e^{2\alpha}),$$

$$d_6 = 0, \quad d'_6 = -\frac{c^2}{40} C_2 e^{6\alpha},$$

and for $n \geqq 4$, $\quad d_{2n} = 0, \quad d'_{2n} = 0,$ $\quad \alpha$ being replaced by α_1.

(*xx*) $\quad \widehat{\alpha\alpha} = 0, \quad \widehat{\alpha\beta} = C_{2m} \cos 2m\beta,$ when $\alpha = \alpha_1$.

We omit the ϕ's and the odd ψ's.

We have then, for $a=a_1$,

$$\psi_{2m-2}=0, \quad \psi'_{2m-2}=\frac{c^2\,C_{2m}}{4\,(2m-1)},$$

$$\psi_{2m}=-\frac{c^2\,C_{2m}}{4m\,(4m^2-1)}\sinh 2a,$$

$$\psi'_{2m}=-\frac{m\,c^2}{4m^2-1}\,C_{2m}\cosh 2a,$$

$$\psi_{2m+2}=0, \quad \psi'_{2m+2}=\frac{c^2\,C_{2m}}{4\,(2m+1)},$$

and $\quad \psi_{2n}=0,\ \psi'_{2n}=0 \quad$ for $n \geqq m+2$ and $n \leqq m-2$.

Solving these equations, we find

$$d_0=0, \quad d_{2m-2}=-\frac{c^2\,C_{2m}}{8\,(2m-1)}\,e^{2ma},$$

$$d'_{2m-2}=\frac{c^2\,C_{2m}}{8\,(2m-1)}\,e^{(2m-2)a},$$

$$d_{2m}=\frac{c^2\,C_{2m}}{8\,(2m+1)}\,e^{2ma},$$

$$d'_{2m}=\frac{c^2\,C_{2m}}{8m\,(4m^2-1)}\,[(2m-1)\,(m+1)\,e^{2a}$$
$$-(2m+1)\,(m-1)\,e^{-2a}]\,e^{2ma},$$

$$d_{2m+2}=0, \quad d'_{2m+2}=-\frac{c^2\,C_{2m}}{8\,(2m+1)}\,e^{(2m+2)a},$$

and $\quad d_{2n}=0,\ d'_{2n}=0 \quad$ for $n \geqq m+2$ and $1 \leqq n \leqq m+2$, a being replaced by a_1.

(*xxi*) $\widehat{aa}=0,\ \widehat{a\beta}=C_3\cos 3\beta$, when $a=a_1$.

We omit the ϕ's and the even ψ's.

We have then, for $a=a_1$,

$$\psi_1\cosh a-\psi'_1\sinh a=-\frac{c^2}{8}\,C_3\sinh a,$$

$$\psi_3=-\frac{c^2}{48}\,C_3\sinh 2a, \qquad \psi'_3=-\frac{3c^2}{16}\,C_3\cosh 2a,$$

$$\psi_5=0, \quad \psi'_5=\frac{c^2}{16}\,C_3,$$

and for $\quad n\geqq 3, \quad \psi_{2n+1}=0, \quad \psi'_{2n+1}=0.$

Solving these equations, we find

$$d_1 = -\frac{c^2}{16} C_3 e^{3\alpha}, \quad d'_1 = \frac{c^2}{16} C_3 e^{\alpha},$$

$$d_3 = \frac{c^2}{32} C_3 e^{3\alpha}, \quad d'_3 = \frac{c^2}{96} C_3 (5e^{5\alpha} - 2e^{\alpha}),$$

$$d_5 = 0, \quad d'_5 = -\frac{c^2}{32} C_3 e^{5\alpha},$$

and for $n \geqq 3$, $d_{2n+1} = 0$, $d'_{2n+1} = 0$, α being replaced by α_1.

(*xxii*) $\widehat{\alpha\alpha} = 0$, $\widehat{\alpha\beta} = C_{2m+1} \cos (2m+1)\beta$, when $\alpha = \alpha_1$.

We omit the ϕ's and the even ψ's.

We have then, when $\alpha = \alpha_1$,

$$\psi_1 \cosh \alpha - \psi'_1 \sinh \alpha = 0,$$

$$\psi_{2m-1} = 0, \quad \psi'_{2m-1} = \frac{c^2 C_{2m+1}}{8m},$$

$$\psi_{2m+1} = -\frac{c^2 C_{2m+1}}{4m(2m+1)(2m+2)} \sinh 2\alpha,$$

$$\psi'_{2m+1} = -\frac{(2m+1) c^2 C_{2m+1}}{4m(2m+2)} \cosh 2\alpha,$$

$$\psi_{2m+3} = 0, \quad \psi'_{2m+3} = \frac{c^2 C_{2m+1}}{8(m+1)},$$

and $\psi_{2n+1} = 0$, $\psi'_{2n+1} = 0$ for $n \geqq m+2$ and $1 \leqq n \leqq m-2$.

Solving these equations, we find

$$d_{2m-1} = -\frac{c^2 C_{2m+1}}{16m} e^{(2m+1)\alpha},$$

$$d'_{2m-1} = \frac{c^2 C_{2m+1}}{16m} e^{(2m-1)\alpha},$$

$$d_{2m+1} = \frac{c^2 C_{2m+1}}{16(m+1)} e^{(2m+1)\alpha},$$

$$d'_{2m+1} = \frac{c^2 C_{2m+1}}{16m(2m+1)(2m+2)} \times \left[2m(2m+3)e^{2\alpha} - (2m-1)(2m+2)e^{-2\alpha}\right] e^{(2m+1)\alpha},$$

$$d_{2m+3} = 0, \quad d'_{2m+3} = -\frac{c^2 C_{2m+1}}{16(m+1)} e^{(2m+3)\alpha},$$

and $d_{2n+1}=0$, $d'_{2n+1}=0$, for $n \geqq m+2$ and $n \leqq m-2$, α being replaced by α_1.

Combining the elementary solutions, (*i*)-(*xxii*), we get the stress function for any distribution of tractions on the boundary $\alpha=\alpha_1$, subject to the condition that the surface tractions form a system of forces in equilibrium. Thus, with the surface tractions (26) on $\alpha=\alpha_1$, the stress function χ is given by (28), (29) and (30), where

$$b_0 = \frac{c^2}{8}\left[2A_0-(A_2-D_2)e^{2\alpha}\right],$$

and for $n \geqq 1$,

$$b_n=\frac{c^2}{8(n+1)}\left[(A_n-D_n)e^{n\alpha}-(A_{n+2}-D_{n+2})e^{(n+2)\alpha}\right];$$

$$c_0 = \frac{c^2}{4}\left[2A_0 \cosh 2\alpha-A_2\right];$$

$$b'_1 = \frac{c^2}{16}\left[k_1(2e^{-2\alpha}+e^{4\alpha})-(3A_3-D_3)e^{\alpha}\right],$$

$$b'_2=-\frac{c^2}{8}A_2+\frac{c^2}{24}\left[(A_2-2D_2)e^{4\alpha}+(2A_4-D_4)e^{2\alpha}\right],$$

and for $n \geqq 3$,

$$\begin{aligned} b'_n=&-\frac{c^2}{8n(n-1)}\left[\{(n-2)A_{n-2}-nD_{n-2}\}e^{n\alpha}\right.\\ &\left.+\{nA_n-(n-2)D_n\}e^{(n-2)\alpha}\right]\\ &+\frac{c^2}{8n(n+1)}\left[\{nA_n-(n+2)D_n\}e^{(n+2)\alpha}\right.\\ &\left.+\{(n+2)A_{n+2}-nD_{n+2}\}e^{n\alpha}\right]; \end{aligned}$$

$$d_0=\frac{c^2}{8}\left[2C_0-(B_2+C_2)e^{2\alpha}\right],$$

and for $n \geqq 1$,

$$d_n=\frac{c^2}{8(n+1)}\left[(B_n+C_n)e^{n\alpha}-(B_{n+2}+C_{n+2})e^{(n+2)\alpha}\right];$$

$$d'_1=\frac{c^2}{16}\left[k_2(e^{4\alpha}-2e^{-2\alpha})+(3B_3+C_3)e^{\alpha}\right],$$

$$\begin{aligned} d'_2=&-\frac{c^2}{8}\left[B_2+2C_0e^{2\alpha}\right]\\ &+\frac{c^2}{24}\left[(B_2+2C_2)e^{4\alpha}+(2B_4+C_4)e^{2\alpha}\right], \end{aligned}$$

and for $n \geqq 3$,

$$d'_n = -\frac{c^2}{8n(n-1)}\left[\{(n-2)B_{n-2}+nC_{n-2}\}e^{n\alpha}\right.$$

$$\left.+\{nB_n+(n-2)C_n\}e^{(n-2)\alpha}\right]$$

$$+\frac{c^2}{8n(n+1)}\left[\{nB_n+(n+2)C_n\}e^{(n+2)\alpha}\right.$$

$$\left.+\{(n+2)B_{n+2}+nC_{n+2}\}e^{n\alpha}\right],$$

α being replaced by α_1.

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4

Lines of Striction on the Quadric and on some other Scrolls.

BY

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1. The line of striction of one system of generators of a general quadric is a rational quartic curve of type (1, 3). This result, which is well-known, was first proved by a *purely geometrical* method by Mr. K. K. Mitra.[1] In this section will be given a somewhat shortened proof by using the theory of correspondence. In §2 will be discussed by purely geometrical methods, several cases wherein the line of striction degenerates. In §3 a new geometrical proof is given which is applicable to many scrolls. A few special scrolls are considered in §4.

We start with the same geometrical principles as Mitra uses. Let P be the point at infinity on a given λ-generator G of the quadric Q, and let R be the point at infinity on the common perpendicular to G and its consecutive generator. We observe that R is the reciprocal of the tangent at P to the conic q at infinity with respect to Ω, the circle at infinity. In other words, P *and* R *are corresponding points on the conic* q *and its reciprocal* q' *with respect to* Ω. This idea is the same as that underlying Mr. Mitra's method, but the present method of expression is more convenient here.

Now if PR meets the quadric again in T, the μ-generator through T meets the generator G at the central point of the latter.[2] To show that the line of striction is a quartic curve, it must be shown that any μ-generator contains three central points of the λ-system. This is equivalent to finding the number of points R on q' such that the line

[1] Bull. Cal. Math. Soc., **XXIV**, *187-198*. More recently, Mr. B. Ramamurti who presented his paper to the ninth conference of the Indian Mathematical Society in 1935 has proved the same result by the method of apolarity.

[2] *Vide* Mitra, *loc. cit.*

joining R to its corresponding point P on q passes through a given point T on q. If T were a general point anywhere, this number would mean the class of the curve formed as the envelope of the lines joining corresponding points of the conic q and its reciprocal q'. By a well-known theorem [1] in the theory of correspondence, this class is equal to the sum of the orders of the two curves minus the number of united points, if any. Hence, in the general case, the class is 4. But when T lies on q, one of the four positions for the point P is T itself. If P_1, P_2, P_3 are the other points, then the central points of the λ-generators through P_1, P_2, P_3 lie on the μ-generator through T.

The proof of the theorem is now complete.

2. *Special cases*:

(*a*) *Quadric of Revolution*: The necessary and sufficient condition that the quadric is one of revolution is that the conic q has double contact with Ω. Let LM be the chord of contact and N its pole. The following lemma can be established by a straight-forward proof, and will be obvious by projection:

If a conic S has double contact with another conic K, the lines joining corresponding points of S and its reciprocal with respect to K are concurrent at the pole of the common chord.

Hence, continuing the notation of § 1, taking any point T on q, let P and R be corresponding points on the conic q and q'. Then the central point C on the λ-generator through P is at its intersection with the μ-generator through T. Since the line PTR passes through N by the lemma, and CPT is the tangent plane at C, it follows that CN is a tangent to the quadric at C. Hence C lies on the polar plane of N. The locus of C is therefore the section of the quadric by the polar plane of N. Now the polar plane of N evidently passes through L and M, since NL and NM are tangents to the quadric. Hence the section is a circle, *viz.*, the central circular section.

The line of striction of either system of generators of a quadric of revolution is a circle, viz., the central circular section.

(*b*) *The general paraboloid*: Mr. Mitra's discussion is incomplete. The conic q now consists of two generators g_λ and g_μ, say which meet at X. Let R_1 and R_2 be the poles of g_λ and g_μ with

[1] H. F. Baker, Principles of Geometry, VI, 10.

respect to Ω. Then the polar plane of R_2 will contain the central points of all the λ-generators, and the polar plane of R_1 will contain the central points of all the μ-generators. It is easy to verify that the polar plane of any point at infinity, *i.e.*, the diametral plane of any line, passes through the point X (X being the "centre" of the paraboloid, this simply means that the diametral plane passes through the centre). Now the section of the paraboloid by any plane through X not containing g_λ or g_μ is a parabola.

Hence the line of striction of either system of generators of a general paraboloid is a parabola.

(c) There is a special case of (b) to be noted. In the general case, R_1 and R_2 will not lie on the quadric. But now suppose that R_2 lies on g_λ, and R_1 lies on g_μ. One of these conditions evidently implies the other. Since R_1 and R_2 are conjugate points with respect to Ω, this means that the generators g_λ and g_μ are now perpendicular, or what is the same thing, any plane through g_λ and any plane through g_μ are perpendicular.

Since R_1 is conjugate to every point on g_μ, the μ-generator through R_1 is perpendicular to all the λ-generators. Similarly, the λ-generator through R_2 is perpendicular to all the μ-generators.

When the section of a paraboloid by the plane at infinity consists of two perpendicular lines, the line of striction of either system of generators is a particular generator of the opposite system. In particular, the generators constituting the lines of striction of the two systems are at right angles.

Conversely, we shall prove that *if the line of striction of one system of generators of a quadric be a generator, not altogether at infinity, belonging to the other system, the quadric is a paraboloid whose section at infinity consists of perpendicular lines.*

Let any λ-generator G meet the plane at infinity in P, and let the line of striction of the λ-generators be a μ-generator meeting the plane at infinity in T. Let the common perpendicular between G and its consecutive generator meet the plane at infinity in R. Then we have seen that PR passes through T and that P and R are corresponding points on a conic and its reciprocal with respect to Ω. Since the μ-generator through T contains the central points of all the λ-generators, this means that whatever point P may be on the conic

at infinity, the line PR passes through a fixed point T. The class of the envelope of the lines joining corresponding points of a curve and its reciprocal is finite if the order of the curve is greater than unity. The only way of the class becoming infinite is when R coincides with T, *i.e.*, when P moves on a straight line, *viz.*, the polar reciprocal of T with respect to Ω. The result now follows.

The bearing of this result with a well-known property of ruled surfaces may be mentioned. If a curve on a scroll is a geodesic as well as the line of striction, it cuts the generators at a constant angle. The result here is that this constant angle is a right angle if the scroll be a quadric and the geodesic a straight line. The nature of the quadric becomes specialised as described.

The above result has another interesting bearing with what is known as the Peterson-Morley Theorem. The theorem is the following: [1]

a, b, c, are three skew lines. a', b', c', are the common perpendiculars of (b, c), (c, a), (a, b). p, q, r are the common perpendiculars of (a, a'), (b, b'), (c, c'). Then p, q, r can be cut at right angles by a line e.

We can now state that *p, q, r lie on a paraboloid whose section at infinity consists of perpendicular lines.*

For if P, Q, R, E be the points at infinity on p, q, r, e, then E is conjugate to P, Q, R with respect to Ω. Hence P, Q, R lie on a line forming a generator at infinity to the quadric determined by p, q, r while E lies on the other generator at infinity. The result follows.

We would be having one more proof of the Peterson-Morley Theorem if the above version of it can be otherwise proved.

(*d*) *The paraboloid of revolution*: The conic at infinity now consists of a pair of tangents g_λ and g_μ to Ω. The poles R_1 and R_2 of g_λ and g_μ are the points of contact of these tangents. Hence the line of striction of the λ-generators is the μ-generator through R_2, *i.e.*, g_μ itself.

The lines of striction of the two systems of generators of a paraboloid of revolution are the generators at infinity. These are of course

[1] *Vide* Proc. Camb. Phil Soc., 30, *102* and *107* (1934). Also Journal Lond. Math. Soc., 11, *24*.

imaginary. There is no difficulty in seeing that this result can be considered as included in case (*a*).

Most of these cases have been previously worked out analytically.[1] I believe that a purely geometrical discussion has not been done before. For case (*c*), the equation of the quadric can be reduced to the form $x^2-y^2=2z$. Sommerville is however incorrect when he writes that the lines of striction are given by $x^2-y^2=0$; $z=0$ and by $x^2-y^2=0$; $w=0$ where $w=0$ is the plane at infinity. Only the latter lines give the lines of striction, otherwise it would mean that the λ-generators, say, have for their lines of striction two μ-generators, in other words, there would be two central points on each generator, which is impossible.

8. In this section, I shall explain a different method of proof for the general case, which is very much shorter than Mr. Mitra's proof or its modification that I have given in § 1.

The line of striction is evidently a rational curve whenever the equations of the generators can be expressed rationally in terms of a parameter.

Now, if P and Q are the points at infinity on two skew lines, neither of which is wholly at infinity, and if R is the pole of PQ with respect to Ω (the circle at infinity), the transversal through R to the two lines is their common perpendicular. Hence the necessary and sufficient condition that the common perpendicular should meet one of the lines (and hence both) at infinity is that PQR is a line, *i.e.*, PQ touches the circle at infinity.

To prove that the line of striction of a general quadric is a quartic curve, we have to show that there are four points at infinity on the curve, *i.e.*, that there are four generators of the system considered such that the common perpendicular between any one of them and its consecutive generator is altogether at infinity. Now, there are four common tangents between Ω and q, where q is the section at infinity of the general quadric. If P be the point of contact with q of one of these tangents, then the generator at P will have its central point at P. The line of striction is therefore a quartic curve which meets the plane at infinity at the points where q touches the common tangents between q and Ω. These points are evidently all imaginary, even if q be real.

[1] Sommerville, Analytical Geometry of Three Dimentions, *305*.

It also follows that *the lines of striction of the two systems of generators have the same points at infinity.*

Applying this method to the quadric of revolution, we see at once that the line of striction is a circle, since there are now only two common tangents between Ω and q, and the points of contact are on Ω. The method of § 2 however gives more information.

The case of the paraboloid cannot be treated by this method.

4. The method of §3 appears to be applicable for any scroll which does not present peculiarities at infinity. The class of the section at infinity could be obtained by the aid of Plücker's equations, and hence the number of common tangents between the section and Ω could be found. Care, however, may be necessary when the multiple points[1] on the section at infinity are such that one or more of the tangents there to the section also happen to touch Ω. An alternative method, such as that described for the quadric, or an analytical method, though longer, will serve a useful purpose as a check on the value given by the general geometrical method. We shall now consider one such method for a few special scrolls.

The cubic scroll of the first type.[2] This surface has two directrix lines, one of which is the double line. Let us consider a point O on the double line, and not lying on the line of striction. Two generators OA and OB pass through this point, and their plane contains the simple directrix. Now a generator in its general position can intersect the line of striction in one point only, *viz.*, at the central point of the generator; for, if it contained the central point of any other generator, the two generators meet there, and therefore the point would lie on the double line. The order of the line of striction is therefore equal to $r+2$ where r is the number of points in which the line of striction cuts the simple directrix. Let O be one of these points, P the point at infinity on the generator through O, R the point at infinity on the common perpendicular between the generator through O and its consecutive generator. As in §1, P and R are conjugate points with respect to Ω, and are corresponding points on the section of the scroll

[1] Multiple points will necessarily exist when the order of the scroll is greater than two, since such a scroll always possesses a double or multiple curve.

[2] *Vide* Salmon, Analytical Geometry of Three Dimensions, §520.

Reference may also be made to the following papers by the present writer, in which some properties of this scroll are discussed :

Journ. Indian Math. Soc., **XIX**, *44*.

Do. **I** (New Series), *251*.

by the plane at infinity, and its reciprocal with respect to Ω. Let T be the point at infinity on the simple directrix. The tangent plane at C contains CT, CP and CR. Hence PR passes through T. The number of points C is thus equal to the number of points P that we can find at infinity such that PR passes through T, for, given one such point P, the corresponding C is the point of intersection of the generator through P with the simple directrix. One position of P however is at T itself, since T lies on the section of the scroll at infinity, and we have to exclude this point because the condition that T lies on the line of striction is that the tangent at T to the section also touches Ω, which condition may not be satisfied. Hence, the number r is one less than the class of the envelope of the lines joining corresponding points on the section at infinity and its reciprocal. The section at infinity being a nodal cubic in general, its reciprocal is of order 4. Therefore $r=3+4-1=6$. The order of the line of striction is therefore eight. The curve is rational, since the generators can be expressed rationally in terms of a parameter. Hence, *the line of striction of the general cubic scroll of the first type is a rational curve of order eight.*

The general method of §3 gives the same result. The nodal cubic at infinity is of class 4, and hence has eight tangents in common with Ω. The points of contact of these tangents are the points at infinity on the line of striction.

It may however happen that the section at infinity is a cuspidal cubic. This is the case when one of the two unodes of the surface is at infinity. The value of r is now $3+3-1=5$, so that *the line of striction is now a curve of order seven.* To see how this result follows by the other method, we have six common tangents between Ω and the cuspidal cubic. But in addition to the points of contact of these tangents with the cubic, the unode at infinity also lies on the line of striction because the two generators of the scroll through a unode coincide, so that the unodes can be regarded as the central points on the torsal generators. The seven points of intersection of the line of striction with the plane at infinity are thus accounted for.

These results will not hold in special cases when the cubic section at infinity is degenerate or when it has special contact relations with Ω.

The Quartic Scroll (Type II B of Edge; Type VII of Salmon[1]*).* This surface is formed by the chords of a twisted cubic which belong

[1] Salmon, *loc. cit.*, Art 540; W. L. Edge, Theory of Ruled Surfaces, Art. 60.

to a special linear complex, *i.e.*, chords meeting a fixed directrix line.

If A is any point on the cubic and AB, AC the generators through A, then BC is another generator. As in the previous case, the line of striction can meet each of the sides of the triangle ABC in one point only. Since the section at infinity is a trinodal quartic whose reciprocal is a sextic, the number of points in which the line of striction meets the directrix $=4+6-1=9$, using the same argument as for the cubic scroll of the first type. *The order of the line of striction for this type of scroll is, in its general case, therefore equal to twelve.* The same result is obtained by the other method.

The Quartic Scroll (Type IV A of Edge; Type II of Salmon[1]*).* The surface has one triple line and one simple directrix. Through any point of the triple line there pass three generators lying in a plane through the other directrix. The section at infinity is a quartic with a triple point. As long as the tangents at this point to the section are distinct, the class of the section is 6, and the arguments and results for the previous type of scroll continue to hold good.

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[1] Edge, *loc. cit.*, Art. 62; Salmon, *loc. cit.*, Art. 547.

5

SIMULTANE IDENTITATEN

By

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Vorbemerkung : Das Symbol $\overset{n}{=}$ bedeutet die Gleichheit der Potenzsummen gleicher Potenzen in n verschiedenen Exponenten z.B.

$$a,\ b,\ c,\ d \overset{n}{=} e,\ f,\ g,\ h \quad (n = 1,\ 2,\ 3)$$

bedeutet das system

$$\begin{aligned} a+b+c+d &= e+f+g+h, \\ a^2+b^2+c^2+d^2 &= e^2+f^2+g^2+h^2, \\ a^3+b^3+c^3+d^3 &= e^3+f^3+g^3+h^3. \end{aligned}$$

I. Die simultane Identität

$$A_1,\ A_2,\ A_3,\ \dots\dots,\ A_8 \overset{n}{=} B_1,\ B_2,\ B_3,\ \dots\dots,\ B_8 \ (n=1,\ 2,\ 3, \dots,\ 6).$$

Ich habe diese Identität gelöst in meiner Arbeit "On the Equation $A_1^n + A_2^n + \dots\dots = B_1^n + B_2^n + \dots\dots$ and Allied Forms" in "The Mathematics Student", Vol. II, 1934. Hier soll eine andere, bisher unbekannte einfache Lösungsart gezeigt werden.

Besteht die Relation

$$F_1,\ F_2,\ \dots\dots,\ F_z \overset{n}{=} G_1,\ G_2,\ \dots\dots,\ G_z \qquad \text{für } n=1,\ \dots,\ x,$$

wobie x *ungerade* und $z = x+1$ ist, dann gilt, wenn man

$$s = \frac{2}{z}(F_1+F_2+\dots\dots\dots+F_z)$$

setzt, auch die Identität

$$\begin{aligned} &F_1,\ F_2,\ \dots,\ F_z,\ s-G_1,\ s-G_2,\ \dots\dots,\ s-G_z \\ &\quad \overset{n}{=} G_1,\ G_2,\ \dots\dots\ G_z,\ s-F_1,\ s-F_2,\ \dots\dots,\ s-F_z \end{aligned}$$

für $n=1$ bis $x+3$, wie durch die Newtonschen Formeln der Gleichungstheorie bewiesen wird.

Besteht also die Relation

$$H_1, H_2, H_3, H_4 \stackrel{n}{=} K_1, K_2, K_3, K_4, \qquad \text{für } n=1, 2, 3,$$

so muss, wenn man

$$s = \frac{2}{4}(H_1+H_2+H_3+H_4)$$

setzt, auch die Identität

$$H_1, H_2, H_3, H_4, s-K_1, s-K_2, s-K_3, s-K_4$$
$$\stackrel{n}{=} K_1, K_2, K_3, K_4, s-H_1, s-H_2, s-H_3, s-H_4$$

$(n=1, 2, \dots, 6)$ gelten.

Damit ist die simultane Identität

$$A_1, A_2, \dots\dots, A_8 \stackrel{n}{=} B_1, B_2, \dots\dots, B_8 \quad (n=1, 2, \dots\dots, 6)$$

gelöst.

Exempel : Aus

$$1, 10, 12, 28 \stackrel{n}{=} 3, 5, 16, 22 \qquad \text{für } n = 1, 2, 3,$$

folgt, da $x = 3$ ungerade und

$$s = \frac{2}{4}(1+10+12+28) = 28$$

ist, das Resultat

$$1, 10, 12, 28, 28-3, 28-5, 28-16, 28-22$$
$$\stackrel{n}{=} 3, 5, 16, 22, 28-1, 28-10, 28-12, 28-28 \quad (n = 1, 2, \dots, 6).$$

II. Besteht die Relation

$$F_1, F_2, \dots\dots, F_s \stackrel{n}{=} G_1, G_2, \dots\dots, G_s \quad (n = 1, 2, \dots\dots, x),$$

wobei x *gerade* und $s = x+1$ ist, dann gilt, wenn man

$$s = \frac{2}{s}(F_1+F_2+\dots+F_s)$$

setzt, nach den Newtonschen Formeln der Gleichungstheorie auch die Identität

$$F_1, F_2, \dots\dots, F_s, s-F_1, s-F_2, \dots\dots, s-F_s$$
$$\stackrel{n}{=} G_1, G_2, \dots\dots, G_s, s-G_1, s-G_2, \dots\dots, s-G_s,$$

für $n = 1$ bis $x+3$.

Daraus folgt für die Relation

$$L_1, L_2, \ldots\ldots, L_5 \overset{n}{=} M_1, M_2, \ldots\ldots, M_5,$$

für $n = 1, 2, 3, 4$, wenn man

$$s = \frac{2}{5}(L_1 + L_2 + \ldots\ldots + L_5)$$

setzt, auch die Gültigkeit der Identität

$$L_1, L_2, \ldots\ldots, L_5, s-L_1, s-L_2, \ldots\ldots s-L_5$$

$$\overset{n}{=} M_1, M_2, \ldots\ldots, M_5, s-M_1, s-M_2 \ldots, s-M_5, \quad (n = 1, 2, \ldots, 7).$$

Damit ist die Identität

$$C_1, C_2, \ldots\ldots, C_{10} \overset{n}{=} D_1, D_2, \ldots\ldots, D_{10}$$

$(n = 1, 2, \ldots\ldots, 7)$ gelöst.

Exempel: Aus

$$1, 13, 14, 30, 32 \overset{n}{=} 2, 8, 21, 25, 34$$

$(n = 1, 2, 3, 4)$ folgt, da x gerade und

$$s = \frac{2}{5}(1 + 13 + 14 + 30 + 32) = 36$$

ist, die Identität

$$1, 13, 14, 30, 32, 36-1, 36-13, 36-14, 36-30, 36-32$$

$$\overset{n}{=} 2, 8, 21, 25, 34, 36-2, 36-8, 36-21, 36-25, 36-34,$$

für $n = 1, 2, \ldots\ldots 7$.

Bemerkt sei, dass es sich auch hier nicht um eine erstmalige Lösung der simultanen Identität

$$C_1, C_2, \ldots\ldots, C_{10} \overset{n}{=} D_1, D_2, \ldots\ldots, D_{10} \qquad (n = 1, 2. \ldots\ldots, 7)$$

handelt, sondern nur um eine von der *bisherigen* Lösungsweise *abweichende* Lösungsart.

III. Wenn ich nun noch eine einfache algebraische Lösung des Systems

$$X, Y, Z \overset{n}{=} U, V, W \qquad (n = 2 \text{ und } 3)$$

anführe, so nur deswegen, weil dessen Lösung in ganzen positiven

Zahlen bisher für unmöglich gehalten wurde. Ich setze

$$X = 8p^2 + 24p + 2\ ; \qquad U = 8p^2 + 72p + 880\ ;$$

$$Y = 9p^2 + 108p + 770\ ; \qquad V = 9p^2 + 120p + 880\ ;$$

$$Z = 12p^2 + 180p + 672\ ; \qquad W = 12p^2 + 204p + 804.$$

$p = 2$ ergibt

$$81,\ 540,\ 571 \overset{n}{=} 271,\ 831,\ 660 \qquad (n = 2 \text{ und } 3).$$

Einen Sonderfall, nämlich

$$1,\ 885,\ 880 \overset{n}{=} 108,\ 108,\ 482 \qquad (n = 2,\ 3)$$

bekommt man, wenn man $p=0$ oder $p=-10$ setzt. Dass mit der genannten Formel nicht *alle* Lösungen des vorgelegten Systems gegeben sind, bedarf kaum besonderer Erwähnung. Es sollte hier nur die *Lösungsmöglichkeit* gezeigt werden.

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Review

Introduction Mathematique aux theories Quantiques—par Gaston Julia (60 fr. Gauthier Villars, Paris).

M. Julia is a master of lucid exposition. Here is one of his books which justifies this statement. A glance through the introductory chapter will enable one to understand the programme of the author. This chapter although historical, is extremely instructive ; it is not a mere statement of facts ; the example with which the author begins brings out clearly the idea and importance of *eigenvalue* of parameters in the problems of mathematical physics. As the reader proceeds, he is introduced to the notion of integral equations and its analogy with linear equations which naturally leads to the idea of Hilbertspace. The aim of the author is to develop these ideas in a systematic way and this is done in the following chapters. The introduction of linear homogeneous geometry (affine geometry) and its metric is effected with such simplicity as to present no difficulty to a beginner, while the chapters follow each other in elegant sequence. There are other books on the same topics, such as that of Weyl, specially meant for the reader of quantum theory ; but the authors of those books seemed to have aimed more at elegance than at simplicity. For this reason this book of M. Julia will attract a wider range of readers than the professional mathematicians. It is, therefore, pleasing that M. Julia has undertaken the task of writing a series of books which would be a continuation of this volume and we are sure that the series would be a boon to the readers of quantum theory.

B. R.

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On Linear Sub-spaces in an Euclidean Hyperspace.

By

N. N. Ghosh.

This paper is written, in continuation of a previous* one, to illustrate the application of the *scalar determinants* in geometrical investigations. The properties of these determinants alone, enable us to build up a kind of vector algebra, applicable to the geometry of hyperspace. The mode of treatment is necessarily elementary and the topics chosen are simple, being restricted to linear sub-spaces in Euclidean hyper-space. Although, I have avoided standard tensor notations, it will be perceived that further developments will merge into the wider tensor calculus. It is believed that the process is new, ensuring greater elegance and expressiveness in the results than are obtained otherwise.

Let a_i ($i = 1, 2, \dots m$) represent m vectors, linearly independent, in an n-dimensional Euclidean space, referred to a system of n rectangular axes. The equation of an m-dimensional linear sub-space A_m is then of the form

$$\rho = a + \sum_{i=1}^{m} x_i a_i , \tag{1}$$

where a is the vector from the origin to a given point on the sub-space and x_i are arbitrary scalar variables.

Associated with (1), we have the non-vanishing scalar determinant of mth order, symbolically represented by

$$\Delta_{a_m} = \begin{Bmatrix} a_1 \; a_2 \; \dots \; a_m \\ a_1 \; a_2 \; \dots \; a_m \end{Bmatrix}, \tag{2}$$

which gives the square of the content of the m-dimensional parallelotope bounded by the vectors a_i as conterminous edges.

Denoting the components of a_i by $(a_{i1}, a_{i2}, \dots a_{in})$, we can express (2) in the ordinary form of a determinant, having for its $(i\text{-}j)$th

* 'On a class of determinants having geometrical applications,' *this Bulletin*, **XXVIII**, *1-12*.

element, the scalar product of a_i and a_j. This determinant Δ_{a_m} will be called the *dominant* of A_m.

The equation (1) shows that for all points on the linear m-space, we must have

$$\begin{Bmatrix} \rho-a \ a_1 \ a_2 \dots a_m \\ \rho-a \ a_1 \ a_2 \dots a_m \end{Bmatrix} = 0. \tag{3}$$

To determine x_i, corresponding to a given point ρ on A_m, we substitute $\rho - a - \sum_{i=1}^{m} x_i a_i$ for a_i in Δ_{a_m}; we have then

$$0 = \begin{pmatrix} a_i \\ \rho-a \end{pmatrix} \Delta_{a_m} + x_i \Delta_{a_m}, \tag{4}$$

where the symbol $\begin{pmatrix} a_i \\ \rho-a \end{pmatrix} \Delta_{a_m}$ represents the scalar determinant Δ_{a_m} in which a_i in the first row is replaced by $\rho-a$.

2. *Parallel and perpendicular vectors to a linear sub-space.*

Let b_j be a vector not belonging to the sub-space A_m. Then we can form a new vector

$$\bar{a}_j = \sum_{i=1}^{m} a_i \begin{pmatrix} a_i \\ b_j \end{pmatrix} \Delta_{a_m} \Big/ \Delta_{a_m}, \tag{5}$$

which evidently belongs to A_m and is the *projection* of b_j on A_m.

If b_j is parallel to A_m, we then have

$$\sum_{i=1}^{m} a_i \begin{pmatrix} a_i \\ b_j \end{pmatrix} \Delta_{a_m} = b_j \Delta_{a_m}, \tag{6}$$

or,

$$\bar{a}_j = b_j.$$

The vector $b'_j = b_j - \bar{a}_j$ is normal to the sub-space A_m, for, taking any vector c_k, we have

$$\begin{Bmatrix} b'_j \\ c_k \end{Bmatrix} = \begin{Bmatrix} b_j \\ c_k \end{Bmatrix} - \begin{Bmatrix} \bar{a}_j \\ c_k \end{Bmatrix},$$

where $\begin{Bmatrix} b'_j \\ c_k \end{Bmatrix}$ is proportional to the cosine of the angle between b'_j

and c_k. By (5), we have then

$$\Delta_{a_m}\begin{Bmatrix} b'_j \\ c_k \end{Bmatrix} = \Delta_{a_m}\begin{Bmatrix} b_j \\ c_k \end{Bmatrix} - \sum_{i=1}^{m}\begin{Bmatrix} a_i \\ c_k \end{Bmatrix}\begin{pmatrix} a_i \\ b_j \end{pmatrix}\Delta_{a_m},$$

$$= \begin{Bmatrix} b_j\ a_1\ a_2 \ldots a_m \\ c_k\ a_1\ a_2 \ldots a_m \end{Bmatrix}, \tag{7}$$

and this vanishes if c_k is parallel to A_m, that is, linearly connected with the vectors a_i.

If $c_k = \bar{a}_k$, we have

$$\begin{Bmatrix} b_j \\ \bar{a}_k \end{Bmatrix} = \begin{Bmatrix} \bar{a}_j \\ \bar{a}_k \end{Bmatrix}. \tag{8}$$

Taking another normal $b'_k = b_k - \bar{a}_k$, it is seen, further,

$$\Delta_{a_m}\begin{Bmatrix} b'_j \\ b'_k \end{Bmatrix} = \begin{Bmatrix} b_j\ a_1\ a_2 \ldots a_m \\ b_k\ a_1\ a_2 \ldots a_m \end{Bmatrix}. \tag{9}$$

The condition that b_j is perpendicular to A_m is

$$\bar{a}_j = 0, \quad i.e., \quad \begin{pmatrix} a_i \\ b_j \end{pmatrix}\Delta_{a_m} = 0. \tag{10}$$

Let ν be the perpendicular on A_m from a given point γ, then putting $b_j = \alpha - \gamma$, in (5), we get

$$\nu = b_j - \bar{a}_j = \alpha - \gamma - \sum_1^m a_i \begin{pmatrix} a_i \\ \alpha - \gamma \end{pmatrix}\Delta_{a_m} \Big/ \Delta_{a_m}, \tag{11}$$

and by (9)

$$\Delta_{a_m}\begin{Bmatrix} \nu \\ \nu \end{Bmatrix} = \begin{Bmatrix} \alpha-\gamma\ a_1\ a_2 \ldots a_m \\ \alpha-\gamma\ a_1\ a_2 \ldots a_m \end{Bmatrix}, \tag{12}$$

which gives the length* of the perpendicular.

8. *A set of vectors mutually perpendicular to one another on the sub-space* A_m.

Let us start with the vector a_1 at the point α. Taking a_2, we replace it by

$$a_2^* = a_2 - a_1\begin{Bmatrix} a_2 \\ a_1 \end{Bmatrix} \Big/ \begin{Bmatrix} a_1 \\ a_1 \end{Bmatrix},$$

* A formula from a different point of view, is given by H. S. Uhler in his paper entitled, 'Least distance from a point to a linear $(n-k)$-space in a linear n-space,' Annals of Mathematics (2), 27, 65-68.

which is perpendicular to a_1, and lies in the plane of a_1 and a_2.

Next, a_3 is replaced by

$$a_3^* = a_3 - \sum_{i=1}^{2} a_i \begin{pmatrix} a_i \\ a_3 \end{pmatrix} \Delta_{a_2} \Big/ \Delta_{a_2},$$

which is perpendicular to the plane of a_1 and a_2. Proceeding in this manner, we finally replace a_m by

$$a_m^* = a_m - \sum_{i=1}^{m-1} a_i \begin{pmatrix} a_i \\ a_m \end{pmatrix} \Delta_{a_{m-1}} \Big/ \Delta_{a_{m-1}}, \tag{13}$$

which is perpendicular to the sub-space A_{m-1} formed by the vectors $a_1, a_2, \ldots a_{m-1}$.

Now $a_1, a_2^*, a_3^*, \ldots a_m^*$ belong to a system of m mutually perpendicular vectors of the sub-space A_m and k of these vectors determine a linear sub-space of k dimensions, while the remaining $m-k$ vectors determine one of $(m-k)$ dimensions, intersecting at the point a, such that every line through the common point in any one of them is perpendicular to every line in the other through the same point. The two linear sub-spaces are *completely orthogonal*.

From (13), we have

$$\Delta_{a_{m-1}} \begin{Bmatrix} a_m^* \\ a_m \end{Bmatrix} = \Delta_{a_{m-1}} \begin{Bmatrix} a_m^* \\ a_m^* \end{Bmatrix} = \Delta_{a_m},$$

and consequently the dominant Δ_{a_m} of A_m, may be expressed in the canonical form

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \begin{Bmatrix} a_2^* \\ a_2^* \end{Bmatrix} \cdots \begin{Bmatrix} a_m^* \\ a_m^* \end{Bmatrix}. \tag{14}$$

Again,

$$\Delta_{a_{m-1}} \begin{Bmatrix} a_m^* \\ b_j \end{Bmatrix} = \begin{Bmatrix} a_m\, a_1 \ldots a_{m-1} \\ b_j\, a_1 \ldots a_{m-1} \end{Bmatrix} = \begin{pmatrix} a_m \\ b_j \end{pmatrix} \Delta_{a_m}.$$

Therefore

$$\begin{Bmatrix} a_m^* \\ a_m \end{Bmatrix} \begin{pmatrix} a_m \\ b_j \end{pmatrix} \Delta_{a_m} = \begin{Bmatrix} a_m^* \\ b_j \end{Bmatrix} \Delta_{a_m}. \tag{15}$$

4. *Normals related to two given linear sub-spaces.*

Let the sub-spaces A_m and B_l be given by the equations

$$\left.\begin{aligned} \rho &= \alpha + \sum_{i=1}^{m} x_i a_i \\ \rho &= \beta + \sum_{i=1}^{l} y_i b_i \end{aligned}\right\}, \tag{16}$$

of m and l dimensions respectively $(m \geq l)$. If the vectors a_i, b_i be all linearly independent of one another, we have, associated with the two sub-spaces, a non-vanishing scalar determinant of $(m+l)$th order

$$\Delta_{a_m b_l} = \begin{Bmatrix} a_1\ a_2 \ldots a_m\ b_1\ b_2 \ldots b_l \\ a_1\ a_2 \ldots a_m\ b_1\ b_2 \ldots b_l \end{Bmatrix}, \quad m+l < n. \tag{17}$$

This will be called the *compound dominant* of A_m and B_l.

Replacing b_j, by $b'_j = b_j - \bar{a}_j$, we have

$$\Delta_{a_m b_l} = \Delta_{a_m} \begin{Bmatrix} b'_1\ b'_2 \ldots b'_l \\ b'_1\ b'_2 \ldots b'_l \end{Bmatrix}.$$

Representing

$$\begin{Bmatrix} b'_1\ b'_2 \ldots b'_l \\ b'_1\ b'_2 \ldots b'_l \end{Bmatrix} \text{ or } \begin{Bmatrix} b'_1\ b'_2 \ldots b'_l \\ b_1\ \ b_2 \ldots b_l \end{Bmatrix},$$

by Δ'_{b_l}, we get

$$\Delta_{a_m b_l} = \Delta_{a_m} \Delta'_{b_l}. \tag{18}$$

Similarly,

$$\Delta_{a_m b_l} = \Delta'_{a_m} \Delta_{b_l}.$$

The linear sub-space B'_l, formed by the vectors b'_i, may be called the linear *normal-sub-space* to A_m, corresponding to B_l. Similarly A'_m will represent the linear normal sub-space to B_l, corresponding to A_m. Now A'_m and A_m cannot be made up of linearly independent vectors, for $2m > m+l$. Hence, at least, $m-l$ of the normals to B_l, will necessarily be parallel to A_m. To obtain these, we proceed as follows:

Choose a set of l vectors a_1, $a_2, \ldots a_l$ in A_m and form the scalar determinant

$$\Delta^{a_l}_{b_l} = \begin{Bmatrix} a_1\ a_2 \ldots a_l \\ b_1\ b_2 \ldots b_l \end{Bmatrix}. \tag{19}$$

Taking each of the remaining $m-l$ vectors from A_m, let us construct the vectors

$$\pi_s = a_s - \sum_{t=1}^{l} a_t \begin{pmatrix} a_t \\ a_s \end{pmatrix} \Delta_{b_l}^{a_l} \Big/ \Delta_{b_l}^{a_l}, \quad (s = l+1, l+2, \ldots m), \tag{20}$$

which obviously belongs to A_m.

Now it can be easily seen that π_s is perpendicular to B_l, for,

$$\Delta_{b_l}^{a_l} \begin{Bmatrix} \pi_s \\ b_k \end{Bmatrix} = \begin{Bmatrix} a_s & a_1 & a_2 \ldots a_l \\ b_k & b_1 & b_2 \ldots b_l \end{Bmatrix} = 0, \quad k \leq l.$$

Next, we shall prove that if a normal to A_m is parallel to B_l, then there is a normal to B_l which is parallel to A_m.

Let c_j denote any vector not lying either in A_m or B_l. Denote

$$c_j - \sum_{t=1}^{l} a_t \begin{pmatrix} a_t \\ c_j \end{pmatrix} \Delta_{a_m} \Big/ \Delta_{a_m}, \qquad \text{a normal to } A_m,$$

and

$$c_j - \sum_{t=1}^{l} b_t \begin{pmatrix} b_t \\ c_j \end{pmatrix} \Delta_{b_l} \Big/ \Delta_{b_l}, \text{ corresponding normal to } B_l,$$

by $c_j^{(a)}$ and $c_j^{(b)}$ respectively. We have then

$$\Delta_{a_m} \begin{Bmatrix} c_j^{(a)} \\ c_j^{(b)} \end{Bmatrix} = \begin{Bmatrix} c_j & a_1 & a_2 \ldots\ldots a_m \\ c_j^{(b)} & a_1 & a_2 \ldots\ldots a_m \end{Bmatrix},$$

and

$$\Delta_{b_l} \begin{Bmatrix} c_j^{(b)} \\ c_j^{(a)} \end{Bmatrix} = \begin{Bmatrix} c_j & b_1 & b_2 \ldots\ldots b_l \\ c_j^{(a)} & b_1 & b_2 \ldots\ldots b_l \end{Bmatrix},$$

whence the proposition follows.

Since $\begin{Bmatrix} c_j^{(a)} \\ c_j^{(b)} \end{Bmatrix}$ must vanish, it follows, further, that the normals are perpendicular to one another. The two sub-spaces are then just *partially orthogonal.*

5. *Common normal to two linear sub-spaces.*

Let the vector to the extremities of the common normal to the sub-spaces (16), be denoted by ν, then

$$\nu = \alpha - \beta + \sum_{t=1}^{m} x_t a_t - \sum_{t=1}^{l} y_t b_t, \tag{21}$$

and satisfies the relation

$$\begin{pmatrix} a_t \\ \nu \end{pmatrix} \Delta_{a_m b_l} = \begin{pmatrix} b_t \\ \nu \end{pmatrix} \Delta_{a_m b_l} = 0.$$

Therefore, substituting for a_t in (17), the expression (21) for ν, we get

$$\begin{pmatrix} a_t \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_l} + x_t \Delta_{a_m b_l} = 0.$$

Similarly

$$\begin{pmatrix} b_t \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_l} - y_t \Delta_{a_m b_l} = 0.$$

These equations determine the position of the common normal.

The vector ν is given by

$$\Delta_{a_m b_l} (\nu-\alpha+\beta) + \sum_{t=1}^{m} a_t \begin{pmatrix} a_t \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_l}$$

$$+ \sum_{t=1}^{l} b_t \begin{pmatrix} b_t \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_l} = 0. \qquad (22)$$

If we consider a compound linear sub-space of $m+l$ dimensions,

$$\rho = \alpha + \sum_{t=1}^{m} x_t a_t + \sum_{t=1}^{l} y_t b_t ,$$

then the perpendicular from β on it, is given by the same expression as (22).

For the length of the common normal, we have then

$$\Delta_{a_m b_l} \begin{Bmatrix} \nu \\ \nu \end{Bmatrix} = \begin{Bmatrix} \alpha-\beta\ a_1\ a_2 \ldots a_m\ b_1\ b_2 \ldots b_l \\ \alpha-\beta\ a_1\ a_2 \ldots a_m\ b_1\ b_2 \ldots b_l \end{Bmatrix}. \qquad (23)$$

If, however, some of the vectors b_t be parallel to A_m, let the compound dominant be represented by

$$\Delta_{a_m b_k} = \begin{Bmatrix} a_1\ a_2 \ldots a_m\ \ b_1\ b_2 \ldots b_k \\ a_1\ a_2 \ldots a_m\ \ b_1\ b_2 \ldots b_k \end{Bmatrix}, \qquad (24)$$

the mutually independent vectors, b_{k+1}, b_{k+2}, ...b_l being linearly connected with a_t.

Now $$\nu = \alpha-\beta+\sum_{1}^{m} x_t a_t - \sum_{1}^{l} y_t b_t ,$$

Making use of the dominant (24), we have

$$0 = \begin{pmatrix} a_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} + x_i \Delta_{a_m b_k} - \sum_{j=k+1}^{l} y_j \begin{pmatrix} a_i \\ b_j \end{pmatrix} \Delta_{a_m b_k},$$

$$0 = \begin{pmatrix} b_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} - y_i \Delta_{a_m b_k}, \qquad (i=1, 2, \ldots k).$$

The vector ν is given by

$$\Delta_{a_m b_k}(\nu-\alpha+\beta) + \sum_1^m a_i \begin{pmatrix} a_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} + \sum_1^k b_i \begin{pmatrix} b_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k}$$

$$= \sum_{h=1}^{l-k} y_{k+h} \left\{ \sum_{i=1}^{m} a_i \begin{pmatrix} a_i \\ b_{k+h} \end{pmatrix} \Delta_{a_m b_k} - b_{k+h} \Delta_{a_m b_k} \right\}.$$

But b_{k+h} is linearly connected with a_i, therefore by (6), the right-hand side vanishes. Hence

$$\Delta_{a_m b_k} (\nu-\alpha+\beta) + \sum_1^m a_i \begin{pmatrix} a_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} + \sum_1^k b_i \begin{pmatrix} b_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} = 0. \quad (25)$$

It must be noticed that through every point in the sub-space

$$\rho = \beta + \sum_1^k b_i \begin{pmatrix} b_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m b_k} \Big/ \Delta_{a_m b_k} + \sum_{j=k+1}^{l} y_j b_j,$$

there is a common normal.

The two sub-spaces are called $(l-k)/l$ *parallel*.

If, again, all the vectors b_i are linearly connected with a_i, the compound dominant is represented by

$$\Delta_{a_m} = \begin{Bmatrix} a_1 \; a_2 \ldots a_m \\ a_1 \; a_2 \ldots a_m \end{Bmatrix}.$$

The two sub-spaces are now *completely parallel*.

As before

$$\nu = \alpha-\beta + \sum_1^m x_i a_i - \sum_1^l y_i b_i,$$

where $b_1, b_2, \ldots b_l$ are all linearly connected with a_i.

We have now

$$\begin{pmatrix} a_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m} + x_i \Delta_{a_m} - \sum_{j=1}^{l} y_j \begin{pmatrix} a_i \\ b_j \end{pmatrix} \Delta_{a_m} = 0,$$

and ν is given by

$$\Delta_{a_m}(\nu-\alpha+\beta) + \sum_1^m a_i \begin{pmatrix} a_i \\ \alpha-\beta \end{pmatrix} \Delta_{a_m}$$

$$= \sum_{j=1}^{l} y_j \left\{ \sum_{i=1}^{m} a_i \begin{pmatrix} a_i \\ b_j \end{pmatrix} \Delta_{a_m} - b_j \Delta_{a_m} \right\} = 0. \qquad (26)$$

From every point in B_l, there is a common normal.

When ν vanishes, the sub-spaces are said to intersect.

6. *Angle between two linear sub-spaces.*

The angle between A_m and B_l is the same as the angle between the sub-spaces

$$\rho = \sum_1^m x_i a_i, \qquad (i)$$

$$\rho = \sum_1^l y_i b_i, \qquad (ii)$$

passing through the origin.

Projecting b_l on (i) orthogonally, a sub-space of l dimensions is generated in (i), which has for its equation

$$\rho = \sum_1^l \bar{x}_i \bar{a}_i . \qquad (iiii)$$

Now the angle between (*i*) and (*ii*) is the same as the angle between (*i*) and (*iii*). Denoting this angle by θ, we have

$$\cos^2\theta = \begin{Bmatrix} \bar{a}_1 \ \bar{a}_2 \ldots\ldots \bar{a}_l \\ \bar{a}_1 \ \bar{a}_2 \ldots\ldots \bar{a}_l \end{Bmatrix} \Big/ \begin{Bmatrix} b_1 \ b_2 \ldots\ldots b_l \\ b_1 \ b_2 \ldots\ldots b_l \end{Bmatrix}. \qquad (27)$$

It can be proved that

$$\bar{\Delta}_{a_l} = \begin{Bmatrix} \bar{a}_1 \ \bar{a}_2 \ldots\ldots \bar{a}_l \\ \bar{a}_1 \ \bar{a}_2 \ldots\ldots \bar{a}_l \end{Bmatrix} = \begin{Bmatrix} \bar{a}_1 \ \bar{a}_2 \ldots\ldots \bar{a}_l \\ b_1 \ b_2 \ldots\ldots b_l \end{Bmatrix},$$

$$= \frac{1}{\Delta_{a_m}} \sum \begin{pmatrix} a_{i_1} & a_{i_2} & a_{i_l} \\ b_1 & b_2 \ldots b_l \end{pmatrix} \Delta_{a_m} \begin{Bmatrix} a_{i_1} \ a_{i_2} \ldots\ldots a_{i_l} \\ b_1 \ b_2 \ldots\ldots\ldots b_l \end{Bmatrix}, \qquad (28)$$

where the summation extends over all l-combinations $(i_1, i_2 \ldots i_l)$ of the integers, $1, 2, \ldots m$, arranged in natural order.

In the above, the symbol

$$\begin{pmatrix} a_{i_1} a_{i_2} \ldots\ldots a_{i_l} \\ b_1 \;\; b_2 \ldots\ldots b_l \end{pmatrix} \Delta_{a_m}$$

represents the scalar determinant Δ_{a_m} in which a_{i_k} in the first row is replaced by b_k for $k=1, 2, \ldots l$.

Hence $\cos^2\theta$ may be expressed in terms of the elements of (i) and (ii).

In the particular case, when $l=m$, $\cos^2\theta$ is given by

$$\frac{\begin{Bmatrix} a_1 \; a_2 \ldots\ldots a_m \\ b_1 \; b_2 \ldots\ldots b_m \end{Bmatrix} \begin{Bmatrix} a_1 \; a_2 \ldots\ldots a_m \\ b_1 \; b_2 \ldots\ldots b_m \end{Bmatrix}}{\Delta_{a_m} \; \Delta_{b_m}}. \tag{20}$$

When some of the vectors b_i are parallel to A_m, the ratio (27) may be further reduced. Let, for definiteness, $b_{k+1}, \ldots b_l$ be all parallel to A_m, then

$$\cos^2\theta = \begin{Bmatrix} \bar{a}_1 \; \bar{a}_2 \ldots\ldots \bar{a}_k \; b_{k+1} \ldots\ldots b_l \\ \bar{a}_1 \; \bar{a}_2 \ldots\ldots \bar{a}_k \; b_{k+1} \ldots\ldots b_l \end{Bmatrix} \Big/ \begin{Bmatrix} b_1 \; b_2 \ldots\ldots b_l \\ b_1 \; b_2 \ldots\ldots b_l \end{Bmatrix},$$

$$= \begin{Bmatrix} \bar{a}_1 \; \bar{a}_2 \ldots\ldots \bar{a}_k \; b_{k+1} \ldots\ldots b_l \\ b_1 \; b_2 \ldots\ldots b_k \; b_{k+1} \ldots\ldots b_l \end{Bmatrix} \Big/ \Delta_{b_l} \; .$$

Denoting the scalar determinant

$$\begin{Bmatrix} b_{k+1} \; b_{k+2} \ldots\ldots b_l \\ b_{k+1} \; b_{k+2} \ldots\ldots b_l \end{Bmatrix},$$

by $\left(\Delta_{b_k}\right)_\mu$, Δ_{b_l} may be expressed in the form

$$\begin{Bmatrix} \sigma_1 \; \sigma_2 \ldots\ldots \sigma_k \\ b_1 \; b_2 \ldots\ldots b_k \end{Bmatrix} \begin{Bmatrix} b_{k+1} \ldots b_l \\ b_{k+1} \ldots b_l \end{Bmatrix},$$

where

$$\sigma_j = b_j - \sum_{i=1}^{l-k} b_{k+i} \begin{pmatrix} b_{k+i} \\ b_j \end{pmatrix} \left(\Delta_{b_k}\right)_\mu \Big/ \left(\Delta_{b_k}\right)_\mu .$$

Similarly

$$\begin{Bmatrix} \bar{a}_1 \; \bar{a}_2 \ldots \bar{a}_k \; b_{k+1} \ldots b_l \\ b_1 \; b_2 \ldots b_k \; b_{k+1} \ldots b_l \end{Bmatrix},$$

is expressed in the form

$$\left(\Delta_{b_k}\right)_\mu \begin{Bmatrix} \bar{a}_1 & \bar{a}_2 \ldots \bar{a}_k \\ \sigma_1 & \sigma_2 \ldots \sigma_k \end{Bmatrix}.$$

Hence $\cos^2\theta$ is reduced to

$$\begin{Bmatrix} \bar{a}_1 & \bar{a}_2 \ldots \bar{a}_k \\ \sigma_1 & \sigma_2 \ldots \sigma_k \end{Bmatrix} \Bigg/ \begin{Bmatrix} \sigma_1 & \sigma_2 \ldots \sigma_k \\ b_1 & b_2 \ldots b_k \end{Bmatrix}. \tag{30}$$

We have seen in Art. 4, that it is generally possible to find a set of at least $m-l$ vectors in A_m such that each of them is perpendicular to B_l. Let us suppose that $a_{l+1}, a_{l+2}, \ldots a_m$ denote these vectors, then $\cos^2\theta$ in (27) is expressible in the simple form

$$\frac{1}{\Delta_{a_m} \Delta_{b_l}} \begin{pmatrix} a_1 & a_2 \ldots a_l \\ b_1 & b_2 \ldots b_l \end{pmatrix} \Delta_{a_m} \begin{Bmatrix} a_1 & a_2 \ldots a_l \\ b_1 & b_2 \ldots b_l \end{Bmatrix},$$

or

$$\frac{\left(\Delta_{b_l}^{a_l}\right)^2 \left(\Delta_{a_l}\right)_\mu}{\Delta_{a_m} \Delta_{b_l}}, \tag{31}$$

which may be further reduced by expressing Δ_{a_m} as a product of scalar determinants of which one member is the minor $(\Delta_{a_l})_\mu$.

7. *Orthogonality of linear sub-spaces.*

When two sub-spaces A_m and B_l are completely orthogonal it is impossible to form a non-vanishing scalar determinant of the type $\Delta_{b_k}^{a_k}$. When the sub-spaces are not orthogonal, the *maximum* order of such scalar determinants is l, in fact, $\Delta_{b_l}^{a_l}$ may be so chosen as not to vanish. When the sub-spaces are $(l-k)/l$ orthogonal, the *maximum* order is k. Let us represent a typical one by

$$\begin{Bmatrix} a_1 & a_2 \ldots a_k \\ b_1 & b_2 \ldots b_k \end{Bmatrix},$$

and call it the *mixed dominant* of A_m and B_l. There are now $m-k$ mutually independent vectors in A_m perpendicular to B_l and $l-k$

mutually independent vectors in B_l perpendicular to A_m. To determine these vectors, we proceed as follows:

Consider the scalar determinant

$$\begin{pmatrix} b_1\ b_2 \ldots b_k \\ a_1\ a_2 \ldots a_k \end{pmatrix} \Delta_{b_l} = \begin{Bmatrix} a_1\ a_2 \ldots a_k\ b_{k+1} \ldots b_l \\ b_1\ b_2 \ldots b_k\ b_{k+1} \ldots b_l \end{Bmatrix},$$

which we simply denote by $\Delta_{b_l}^{a_k}$. Then the vectors

$$a_s - \sum_{i=1}^{k} a_i \begin{pmatrix} a_i \\ a_s \end{pmatrix} \Delta_{b_l}^{a_k} \Big/ \Delta_{b_l}^{a_k} - \sum_{i=k+1}^{l} b_i \begin{pmatrix} b_i \\ a_s \end{pmatrix} \Delta_{b_l}^{a_k} \Big/ \Delta_{b_l}^{a_k}, \tag{32}$$

$$(s = k+1,\ k+2, \ldots\ldots, m)$$

are all perpendicular to B_l. Moreover, all these $m-k$ vectors belong to A_m, for,

$$\sum_{i=k+1}^{l} b_i \begin{pmatrix} b_i \\ a_s \end{pmatrix} \Delta_{b_l}^{a_k} \Big/ \Delta_{b_l}^{a_k} = 0, \ (s = k+1, \ldots\ldots, m), \tag{33}$$

since every scalar determinant

$$\begin{pmatrix} b_i \\ a_s \end{pmatrix} \Delta_{b_l}^{a_k},$$

involved in it vanishes when expanded by Laplace's theorem. Again, consider the scalar determinant

$$\begin{pmatrix} a_1\ a_2 \ldots a_k \\ b_1\ b_2 \ldots b_k \end{pmatrix} \Delta_{a_m} = \begin{Bmatrix} b_1\ b_2 \ldots b_k\ a_{k+1} \ldots a_m \\ a_1\ a_2 \ldots a_k\ a_{k+1} \ldots a_m \end{Bmatrix},$$

which we simply denote by $\Delta_{a_m}^{b_k}$. Then the $l-k$ vectors

$$b_s - \sum_{i=1}^{k} b_i \begin{pmatrix} b_i \\ b_s \end{pmatrix} \Delta_{a_m}^{b_k} \Big/ \Delta_{a_m}^{b_k} - \sum_{i=k+1}^{m} a_i \begin{pmatrix} a_i \\ b_s \end{pmatrix} \Delta_{a_m}^{b_k} \Big/ \Delta_{a_m}^{b_k}, \tag{34}$$

$$(s = k+1,\ k+2,\ \ldots,\ l)$$

are all perpendicular to A_m and it can be proved that they belong to B_l.

When A_m and B_l are completely parallel, we shall always have

$$\Delta_{b_l}^{a_l} \neq 0,$$

and consequently the sub-spaces can never be orthogonal. When A_m^{11} and B_l are h/l parallel, the *minimum* order of the mixed dominant is h, which we may conveniently represent by

$$\begin{Bmatrix} a_1 & a_2 & \dots & a_h \\ b_1 & b_2 & \dots & b_h \end{Bmatrix} a_i = b_i,$$

and the *highest* degree of orthogonality is $(l-h)/l$.

The $m-h$ mutually independent vectors in A_m perpendicular to B_l are then given by

$$a_s - \sum_{i=1}^{h} a_i \begin{pmatrix} a_i \\ a_s \end{pmatrix} \Delta^{a_h}_{b_l} \Big/ \Delta^{a_h}_{b_l}, \quad (s = h+1, \dots, m), \tag{85}$$

where

$$\Delta^{a_h}_{b_l} = \begin{Bmatrix} a_1 & a_2 & \dots & a_h & b_{h+1} & \dots & b_l \\ a_1 & a_2 & \dots & a_h & b_{h+1} & \dots & b_l \end{Bmatrix}.$$

The $l-h$ mutually independent vectors in B_l perpendicular to A_m are given by

$$b_s - \sum_{i=1}^{h} b_i \begin{pmatrix} b_i \\ b_s \end{pmatrix} \Delta^{b_h}_{a_m} \Big/ \Delta^{b_h}_{a_m}, \quad (s = h+1, \dots, l), \tag{86}$$

where

$$\Delta^{b_h}_{a_m} = \begin{Bmatrix} b_1 & b_2 & \dots & b_h & a_{h+1} & \dots & a_m \\ a_1 & a_2 & \dots & a_h & a_{h+1} & \dots & a_m \end{Bmatrix}.$$

Since $a_i = b_i \quad (i = 1, 2, \dots h)$, the above may be expressed in the form

$$b_s - \sum_{i=1}^{h} a_i \begin{pmatrix} a_i \\ b_s \end{pmatrix} \Delta_{a_m} \Big/ \Delta_{a_m}, \tag{87}$$

$$(s = h+1, h+2, \dots, l).$$

8. *Successive projections and minimal vectors.*

Let b be any vector lying on B_l, then its projection $b^{\bar{a}}$ on A_m is given by

$$b^{\bar{a}} = \sum_{i=1}^{m} a_i \begin{pmatrix} a_i \\ b \end{pmatrix} \Delta_{a_m} \Big/ \Delta_{a_m}.$$

If now $b^{\bar{a}}$ be projected on B_l, we get

$$b^{\bar{a}\bar{b}} = \sum_{i=1}^{l} b_i \binom{b_i}{b^{\bar{a}}} \Delta_{b_l} \Big/ \Delta_{b_l}$$

$$= \sum_{i=1}^{l} \sum_{h=1}^{m} b_i \binom{b_i}{a_h} \Delta_{b_l} \binom{a_h}{b} \Delta_{a_m} \Big/ \Delta_{a_m} \Delta_{b_l}. \quad (38)$$

The vector b is a *minimal vector*, if

$$b^{\bar{a}\bar{b}} = u.b, \quad (39)$$

where $u = \cos^2\phi$, ϕ being the angle between b and A_m.

Representing, for a moment,

$$\sum_{h=1}^{m} \binom{b_i}{a_h} \Delta_{b_l} \binom{a_h}{b} \Delta_{a_m} \Big/ \Delta_{a_m} \Delta_{b_l}$$

by the symbol $\binom{b_i}{b}$, and using for b the expression $\sum_{i=1}^{l} y_i b_i$, we can rewrite (39) in the form

$$\sum_{j=1}^{l} \sum_{i=1}^{l} b_i \binom{b_i}{b_j} y_j = u \sum_{i=1}^{l} y_i b_i,$$

whence
$$u y_i = \sum_{j=1}^{l} \binom{b_i}{b_j} y_j, \quad (i = 1, 2, \ldots, l). \quad (40)$$

The above equations determine the y's of the minimal vectors provided u satisfies the determinantal equation of well-known form,

$$\begin{vmatrix} \binom{b_1}{b_1} - u & \binom{b_1}{b_2} & \ldots & \binom{b_1}{b_l} \\ \binom{b_2}{b_1} & \binom{b_2}{b_2} - u & \ldots & \binom{b_2}{b_l} \\ \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \\ \binom{b_l}{b_1} & \binom{b_l}{b_2} & \ldots & \binom{b_l}{b_l} - u \end{vmatrix} = 0. \quad (41)$$

Putting $u = 0$ in the above, we get a determinant which is the product of two rectangular arrays

$$\begin{vmatrix} \binom{b_1}{a_1}\Delta_{b_l}\Big/\Delta_{b_l} & \binom{b_1}{a_2}\Delta_{b_l}\Big/\Delta_{b_l} & \cdots\cdots & \binom{b_1}{a_m}\Delta_{b_l}\Big/\Delta_{b_l} \\ \binom{b_2}{a_1}\Delta_{b_l}\Big/\Delta_{b_l} & \binom{b_2}{a_2}\Delta_{b_l}\Big/\Delta_{b_l} & \cdots\cdots & \binom{b_2}{a_m}\Delta_{b_l}\Big/\Delta_{b_l} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{b_l}{a_1}\Delta_{b_l}\Big/\Delta_{b_l} & \binom{b_l}{a_2}\Delta_{b_l}\Big/\Delta_{b_l} & \cdots\cdots & \binom{b_l}{a_m}\Delta_{b_l}\Big/\Delta_{b_l} \end{vmatrix}$$

and

$$\begin{vmatrix} \binom{a_1}{b_1}\Delta_{a_m}\Big/\Delta_{a_m} & \binom{a_2}{b_1}\Delta_{a_m}\Big/\Delta_{a_m} & \cdots\cdots & \binom{a_m}{b_1}\Delta_{a_m}\Big/\Delta_{a_m} \\ \binom{a_1}{b_2}\Delta_{a_m}\Big/\Delta_{a_m} & \binom{a_2}{b_2}\Delta_{a_m}\Big/\Delta_{a_m} & \cdots\cdots & \binom{a_m}{b_2}\Delta_{a_m}\Big/\Delta_{a_m} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \binom{a_1}{b_l}\Delta_{a_m}\Big/\Delta_{a_m} & \binom{a_2}{b_l}\Delta_{a_m}\Big/\Delta_{a_m} & \cdots\cdots & \binom{a_m}{b_l}\Delta_{a_m}\Big/\Delta_{a_m} \end{vmatrix}$$

and therefore may be expressed in the form

$$\frac{1}{\Delta_{a_m}\Delta_{b_l}}\sum\begin{pmatrix} b_1 & b_2 & \dots & b_l \\ a_{i_1} & a_{i_2} & \dots & a_{i_l} \end{pmatrix}\Delta_{b_l}\begin{pmatrix} a_{i_1} & a_{i_2} & \dots & a_{i_l} \\ b_1 & b_2 & \dots & b_l \end{pmatrix}\Delta_{a_m}, \qquad (42)$$

where the summation extends over all l-combinations $i_1, i_2, \dots i_l$ (arranged in natural order) of the integers $1, 2, \dots, m$.

Referring to (28), we notice that the value of (42) is $\cos^2\theta$, where θ is the angle between A_m and B_l.

Denoting the l roots of the fundamental equation (41) by $u_1, u_2 \ldots u_l$, it follows therefore

$$u_1 . u_2 \ldots\ldots u_l = \cos^2\theta. \tag{48}$$

If k of the roots vanish, the sub-spaces A_m, B_l are k/l orthogonal.

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7

On the Vector derivation of the Invariants and Centroid formulae for Convex Surfaces.

By

S. N. Roy.

I.

Introduction.

In a previous paper,* Mr. R. C. Bose and the present author took their cue from investigations made earlier of certain interesting properties connected with the three different kinds of centroids defined for a closed convex curve and studied the corresponding properties with reference to four different kinds of centroids defined for a closed convex surface, of course, regular and analytic throughout. Associating with each point of the surface, three different kinds of density, in succession, (1) a density proportional to the Gaussian curvature $\frac{1}{R_1 R_2}$, where R_1 and R_2 are the principal radii of curvature at that point, (2) a density proportional to the mean curvature $\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$, (3) a uniform density, the centroids of the surface thus differently weighted, were called respectively, (1) the Gaussian curvature centroid, (2) the mean curvature centroid, (3) the surface centroid; and the masses of the three such weighted surfaces were called respectively the three invariants of the surface. To the three such centroids was added the volume centroid and to the three invariants of the surface, the volume content, thus making up four centroids and four invariants for the surface.

*R.C.Bose and S.N.Roy, "On the four centroids of a closed convex surface," Bull. Cal. Math. Soc., 27, (1935), *119-146*. Other references to the literature on the subject will be found in this paper.

A (1, 1) correspondence being established between the surface and a unit sphere in the sense that, corresponding to a point P on the surface, a point P′ could be taken on the unit sphere, such that the radius vector from the centre of the sphere to P′ is parallel to the outward-drawn normal to the surface at P with direction-cosines (α, β, γ), it was shown that any characteristic defining the surface could be regarded as a function of position (α, β, γ) on the unit sphere. In particular, H, the perpendicular from any point (taken inside the surface for convenience) to the tangent plane to the surface at P, could be looked upon as a function of position (α, β, γ) at the corresponding point P′ of the unit sphere. This H is commonly known as Minkowski's Stützfunktion.

Grad, etc., were associated with differentiations connected with movement from one point of the unit sphere to a consecutive point on it, all vectors being thus situated on the tangent planes to the unit sphere; and $\mathbf{r}^{\circ}, \mathbf{r}', \mathbf{r}'', \mathbf{r}'''$ denoted the vectors from the origin to the four centroids and 4π, M, S and V, the four surface invariants.

It was shown that

$$4\pi\, \mathbf{r}^{\circ} = 3\int \mathbf{n} H d\omega, \tag{1·1}$$

$$2M\, \mathbf{r}' = \int (3H^2 - \nabla H)\mathbf{n}\, d\omega, \tag{1·2}$$

$$S\, \mathbf{r}'' = \int \{H^3 - H\nabla H + \tfrac{1}{2}(\text{grad}H\ \text{grad})\nabla H\}\mathbf{n}\, d\omega, \tag{1·3}$$

$$4V\, \mathbf{r}''' = \int \{H^4 - 2H^2\nabla H + H(\text{grad}H\ \text{grad})\,\nabla H + \tfrac{1}{2}(\nabla H)^2\}\mathbf{n}\, d\omega$$
$$+ \tfrac{1}{2}\int \{(\text{grad}H\ \text{grad})(\nabla H)\}\text{grad}H\, d\omega, \tag{1·4}$$

where $\nabla H = \text{grad}H.\ \text{grad}H$, $\mathbf{n}$ is the unit vector in the direction of the outward drawn normal to the surface at P, *i. e.*, the radius vector from the centre of the unit sphere to the point P′ on it, $d\omega$ is the element of area on the unit sphere and the single sign of integration stands for the double integral.

The four invariants came out as

$$4\pi = 4\pi, \tag{1·5}$$

$$M = \int H d\omega, \tag{1·6}$$

$$S = \int (H^2 - \tfrac{1}{2}\nabla H) d\omega, \tag{1·7}$$

$$V = \int \{\tfrac{1}{3}H^3 - \tfrac{1}{2}H\nabla H + \tfrac{1}{4}(\text{grad}H\ \text{grad})(\nabla H)\} d\omega. \tag{1·8}$$

The object of the present note is to express the mean radius of curvature $\frac{1}{2}(R_1 + R_2)$ *and the Gaussian radius of curvature* $R_1 R_2$ *purely in terms of vector invariants, the vectors being* gradH *and other vectors derivable from it and to derive the formulae for the different centroids and surface invariants defined above, by*

purely vectorial methods or, to be more explicit, by means of the well-known ∇*-calculus (del-calculus). This considerably simplifies the labour of calculation involved in the previous derivations and like all vector methods, gives an insight into the process of calculation. The interesting geometrical properties connected with these centroids and following directly from the formulae* (1·1) *to* (1·8), *have been proved by the methods of vector differential geometry in the earlier paper which ought to be referred to for these.*

II.

The appropriate vector calculus and the vector expressions for the curvatures.

Any scalar quantity V which is a function of the position (α, β, γ), on the unit sphere can be made a homogeneous function of (α, β, γ) of any desired degree p, by virtue of the relation $\alpha^2+\beta^2+\gamma^2=1$ and after such a transformation, it was shown in the previous paper that *

$$\int \frac{\partial V}{\partial \alpha}\, d\omega = (p+2)\int \alpha V d\omega.$$

If we regard $\dfrac{\partial}{\partial \alpha}$, $\dfrac{\partial}{\partial \beta}$, etc. as simply space differentiations of a function of space coordinates (α, β, γ), then we have

$$\int \operatorname{grad} V d\omega = (p+2)\int \mathbf{n} V d\omega, \qquad (2·1)$$

where $\mathbf{n}$ is the unit radius vector from the centre of the unit sphere. This is, of course, a vector equation. Similarly, if V is a vector function of (α, β, γ), of degree p, then

$$\int \operatorname{div} V d\omega = (p+2)\int \mathbf{n}.V d\omega, \qquad (2·2)$$

$$\int \operatorname{rot} V d\omega = (p+2)\int \mathbf{n}\times V d\omega. \qquad (2·3)$$

Also if W and V are two homogeneous vector functions of α, β, γ and if p be the sum of their degrees, then

$$\int W \operatorname{div} V d\omega = -\int (V \operatorname{grad}) W d\omega + (p+2)\int W.(\mathbf{n}.\, V) d\omega. \qquad (2·4)$$

The formulae (2·1) to (2·4) are the fundamental integration formulae of the present calculus and will be repeatedly used in course of the calculations in sections III and IV.

* R. C. Bose and S. N. Roy, *loc. cit.*, 128, formula (2·8).

It should be remembered here that div, grad, etc., of the present paper are space differentiations and should be distinguished from div, grad, etc., of the previous paper which are surface differentiations. As a matter of fact,

$$\text{grad}V\ (\text{space}) = \text{grad}V\ (\text{surface}) + p\text{n}V, \tag{2·51}$$

where p is the degree of V. We have similar results for div and rot. In fact, if V is a vector function of space coordinates α, β, γ, it can be shown that

$$\text{div}\ V\ (\text{space}) = \text{div}\ V\ (\text{surface}) + p(\text{n}.V). \tag{2·52}$$

Changing Minkowski's Stützfunktion H into a homogeneous function of (α, β, γ) of degree 1 by virtue of the relation $\alpha^2+\beta^2+\gamma^2=1$, and calling

$$\frac{\partial H}{\partial \alpha}=H_1,\ \frac{\partial H}{\partial \beta}=H_2,\ \frac{\partial H}{\partial \gamma}=H_3,\ \frac{\partial^2 H}{\partial \alpha^2}=H_{11},\ \frac{\partial^2 H}{\partial \beta^2}=H_{22},$$

$$\frac{\partial^2 H}{\partial \gamma^2}=H_{33},\ \frac{\partial^2 H}{\partial \alpha\partial \beta}=H_{12},\ \frac{\partial^2 H}{\partial \beta\partial \gamma}=H_{23},\ \frac{\partial^2 H}{\partial \gamma\partial \alpha}=H_{31},$$

we easily see that*

$$\alpha H_1+\beta H_2+\gamma H_3 = H,$$

or vectorially, $$\text{n}.\ \text{grad}\,H\ (\text{space}) = H. \tag{2·53}$$

Again
$$\left.\begin{aligned}\alpha H_{11}+\beta H_{12}+\gamma H_{13} &= 0\\ \alpha H_{21}+\beta H_{22}+\gamma H_{23} &= 0\\ \alpha H_{31}+\beta H_{32}+\gamma H_{33} &= 0\end{aligned}\right\}, \tag{2·540}$$

or vectorially, $$\text{n}.\ \text{grad}(\nabla H)\ (\text{space}) = 0, \tag{2·54}$$

and also $$(\text{n}.\ \text{grad})\ (\text{grad}\,H)\ (\text{space}) = 0. \tag{2·545}$$

Again, remembering the relations (2·64) to (2·72) of the previous paper, we have

$$\nabla H\ (\text{space}) = \nabla H\ (\text{surface}) + H^2, \tag{2·55}$$

and $$(\text{grad}\,H.\ \text{grad})\ (\text{function of degree } p)\ (\text{space})$$
$$= (\text{grad}\,H.\ \text{grad})\ (\text{function of degree } p)\ (\text{surface})$$
$$+ p^2 H\ (\text{function of degree } p), \tag{2·56}$$

* W. Blaschke, Kreis und Kugel, 139.

$$\tfrac{1}{2}(R_1+R_2) = \tfrac{1}{2}(H_{11}+H_{22}+H_{33}) = \tfrac{1}{2}\,\text{div grad}H.$$

Also* $$R_1R_2 = \begin{vmatrix} H_{11} & H_{12} & H_{13} & \alpha \\ H_{21} & H_{22} & H_{23} & \beta \\ H_{31} & H_{32} & H_{33} & \gamma \\ \alpha & \beta & \gamma & 0 \end{vmatrix}$$

$$=\alpha[\alpha(H_{22}H_{33}-H_{23}^2)+\beta(H_{23}H_{31}-H_{21}H_{33})+\gamma(H_{31}H_{32}-H_{22}H_{31})]$$
$$+\beta[\text{similar terms}]+\gamma[\text{similar terms}]$$
$$=\alpha[\alpha(H_{22}H_{33}-H_{23}^2)+\{(-\alpha H_{13}-\gamma H_{33})H_{31}+(\alpha H_{11}+\gamma H_{31})H_{33}\}$$
$$+\{(-\alpha H_{12}-\beta H_{22})H_{21}+(\alpha H_{11}+\beta H_{21})H_{22}\}]$$
$$+\beta[\text{similar terms}]+\gamma[\text{similar terms}], \text{ from } (2\cdot540),$$
$$=\alpha[\alpha(H_{22}H_{33}-H_{23}^2)+\alpha(H_{11}H_{33}-H_{13}^2)+\alpha(H_{22}H_{11}-H_{21}^2)]$$
$$+\beta[\text{similar terms}]+\gamma[\text{similar terms}]$$
$$=\alpha^2[(H_{22}H_{33}-H_{23}^2)+(H_{11}H_{33}-H_{31}^2)+(H_{22}H_{11}-H_{12}^2)]$$
$$+\beta^2[\text{similar terms}]+\gamma^2[\text{similar terms}].$$

It easily follows from the perfect symmetry of the coefficient of α^2 that β^2 and γ^2 will also have the same coefficient. Therefore

$$R_1R_2 = (H_{22}H_{11}-H_{21}^2)+(H_{22}H_{33}-H_{23}^2)+(H_{33}H_{11}-H_{31}^2)$$
$$=\frac{\partial}{\partial\alpha}(H_1H_{22})-\frac{\partial}{\partial\beta}(H_1H_{12})+\frac{\partial}{\partial\beta}(H_2H_{33})-\frac{\partial}{\partial\gamma}(H_2H_{23})$$
$$+\frac{\partial}{\partial\gamma}(H_3H_{11})-\frac{\partial}{\partial\alpha}(H_3H_{31})$$
$$=\frac{\partial}{\partial\beta}(H_2H_{11})-\frac{\partial}{\partial\alpha}(H_2H_{12})+\frac{\partial}{\partial\gamma}(H_3H_{22})-\frac{\partial}{\partial\beta}(H_3H_{32})$$
$$+\frac{\partial}{\partial\alpha}(H_1H_{33})-\frac{\partial}{\partial\gamma}(H_1H_{31})$$
$$=\tfrac{1}{2}\Big[\frac{\partial}{\partial\alpha}\{H_1(\text{div grad}H-H_{11})\}+\frac{\partial}{\partial\beta}\{H_2(\text{div grad}H-H_{22})\}$$
$$+\frac{\partial}{\partial\gamma}\{H_3(\text{div grad}H-H_{33})\}\Big]$$
$$-\tfrac{1}{4}\left\{\frac{\partial^2}{\partial\beta^2}(H_1^2)+\frac{\partial^2}{\partial\gamma^2}(H_2^2)+\frac{\partial^2}{\partial\alpha^2}(H_3^2)\right\}$$
$$-\tfrac{1}{4}\left\{\frac{\partial^2}{\partial\alpha^2}(H_2^2)+\frac{\partial^2}{\partial\beta^2}(H_3^2)+\frac{\partial^2}{\partial\gamma^2}(H_1^2)\right\}$$

* W. Blaschke, *loc. cit.*, 140.

$= \frac{1}{2}\,\mathrm{div}(\mathrm{grad}\,H\ \mathrm{div\ grad}\,H) - \frac{1}{2}\,\mathrm{div\ grad}\,\nabla H.$

We have thus

$$R_1 + R_2 = \mathrm{div\ grad}\,H, \tag{2·6}$$

and

$$R_1 R_2 = \tfrac{1}{2}\,\mathrm{div}\{\mathrm{grad}H\ \mathrm{div\ grad}H - \tfrac{1}{2}\,\mathrm{grad}\,\nabla H\}. \tag{2·7}$$

III.

The invariants of the surface.

The first invariant 4π need not, of course, be calculated.

The second invariant

$$\begin{aligned} M &= \tfrac{1}{2}\int (R_1 + R_2)\, d\omega \\ &= \tfrac{1}{2}\int \mathrm{div\ grad}\,H\, d\omega, \qquad \text{from (2·6)}, \\ &= \tfrac{1}{2}\int \mathbf{n}.\,\mathrm{grad}\,H\, d\omega, \qquad \text{from (2·2)}, \\ &= \tfrac{1}{2}\int H\, d\omega, \qquad \text{from (2·53)}. \end{aligned} \tag{3·1}$$

The third invariant

$$\begin{aligned} S = \int R_1 R_2 d\omega &= \tfrac{1}{2}\int \{\mathrm{div}(\mathrm{grad}\,H\ \mathrm{div\ grad}H) - \tfrac{1}{2}\mathrm{div\ grad}\nabla H\} d\omega, \\ &\hspace{18em} \text{from (2·7)}, \\ &= \tfrac{1}{2}\int (\mathbf{n}.\mathrm{grad}H\ \mathrm{div\ grad}H) d\omega - \tfrac{1}{4}\int \mathbf{n}.\mathrm{grad}\,\nabla H d\omega, \\ &\hspace{18em} \text{from (2·2)} \\ &= \tfrac{1}{2}\int (H\ \mathrm{div\ grad}H) d\omega, \text{ from (2·53) and (2·54)}, \\ &= \tfrac{3}{2}\int \mathbf{n}.H\ \mathrm{grad}H d\omega - \tfrac{1}{2}\int \nabla H d\omega, \text{ from (2·2) and} \\ &\hspace{14em} \text{ordinary del-calculus}, \\ &= \tfrac{1}{2}\int (3H^2 - \nabla H) d\omega \ \text{(space)}, \ \text{from (2·53)}, \qquad (3·2) \\ &= \int (H^2 - \tfrac{1}{2}\nabla H) d\omega \ \text{(surface)}, \ \text{from (2·55)}. \qquad (3·3) \end{aligned}$$

The fourth invariant

$$V = \tfrac{1}{3}\int H R_1 R_2 d\omega.$$

But

$$\begin{aligned} \int H R_1 R_2 d\omega = \tfrac{1}{2}\int &H\mathrm{div}(\mathrm{grad}H\ \mathrm{div\ grad}H) d\omega \\ &- \tfrac{1}{4}\int H\ \mathrm{div\ grad}\nabla H\, d\omega, \qquad \text{from (2·7)}, \end{aligned}$$

$$= \tfrac{1}{4}\int \operatorname{div}(\operatorname{grad}H^2 \operatorname{div}\operatorname{grad}H)d\omega - \tfrac{1}{2}\int \nabla H \operatorname{div}\operatorname{grad}H d\omega$$

$$-\tfrac{1}{4}\int \operatorname{div}(H \operatorname{grad}\nabla H)d\omega + \tfrac{1}{4}\int \operatorname{grad}H \operatorname{grad}\nabla H d\omega,$$

from ordinary del-calculus,

$$= \tfrac{1}{4}\int \mathbf{n}.\operatorname{grad}H^2 \operatorname{div}\operatorname{grad}H d\omega - \tfrac{1}{2}\int \nabla H \operatorname{div}\operatorname{grad}H d\omega$$

$$-\int H.\mathbf{n}.\operatorname{grad}\nabla H\, d\omega + \tfrac{1}{4}\int \operatorname{grad}H.\operatorname{grad}\nabla H d\omega,$$

from (2·2),

$$= \int H^2 \operatorname{div}\operatorname{grad}H d\omega - \tfrac{1}{2}\int \nabla H \operatorname{div}\operatorname{grad}H d\omega$$

$$+ \tfrac{1}{4}\int \operatorname{grad}H.\operatorname{grad}\nabla H d\omega,$$

from (2·53) and (2·54),

$$= 4\int H^3 d\omega - 2\int H\nabla H d\omega - \int H\nabla H d\omega$$

$$+ \tfrac{1}{2}\int \operatorname{grad}H \operatorname{grad}\nabla H\, d\omega + \tfrac{1}{4}\int (\operatorname{grad}H \operatorname{grad}\nabla H)d\omega,$$

from (2·2),

$$= 4\int H^3 d\omega - 3\int H\, \nabla H d\omega + \tfrac{3}{4}\int \operatorname{grad}H \operatorname{grad}\nabla H d\omega.$$

Therefore

$$V = \int\{\tfrac{4}{3}H^3 - H\nabla H + \tfrac{1}{4}\operatorname{grad}H.\operatorname{grad}\nabla H\}d\omega \quad \text{(space)} \tag{3·4}$$

$$= \tfrac{1}{3}\int\{H^3 - \tfrac{3}{2} H\nabla H + \tfrac{3}{4}\operatorname{grad}H \operatorname{grad}\nabla H\}d\omega \quad \text{(surface)}, \tag{3·5}$$

by (2·53) to (2·56).

IV.

The four Centroids.

$$4\pi\mathbf{r}^\circ = \int \operatorname{grad}H.d\omega$$

$$= 3\int \mathbf{n}.H d\omega, \qquad \text{from (2·1)}. \tag{4·1}$$

$$2M\mathbf{r}' = 2\int \operatorname{grad}H \operatorname{div}\operatorname{grad}H d\omega$$

$$= -2\int (\operatorname{grad}H \operatorname{grad})(\operatorname{grad}H)d\omega + 4\int \operatorname{grad}H(\mathbf{n}.\operatorname{grad}H)d\omega,$$

from (2·4).

It is easily seen from the ordinary del-calculus that

$$(\operatorname{grad}H \operatorname{grad})(\operatorname{grad}H) = \tfrac{1}{2}\operatorname{grad}\nabla H,$$

where $\nabla H = \operatorname{grad}H.\operatorname{grad}H.$

This result will be repeatedly used in course of the calculations.

Therefore, the above expression

$$= -\int \mathrm{grad}(\nabla \mathrm{H})\, d\omega + 4\int \mathrm{H}\,\mathrm{grad}\mathrm{H}\, d\omega, \text{ from } (2{\cdot}53),$$

$$= -\int \mathbf{n}.\nabla \mathrm{H}\, d\omega + 4\int \mathbf{n}.\,\mathrm{H}^2\, d\omega, \quad \text{from } (2{\cdot}1),$$

$$= \int \mathbf{n}.(4\mathrm{H}^2 - \nabla \mathrm{H})\, d\omega \quad \text{(space)}, \tag{4·2}$$

$$= \int (3\mathrm{H}^2 - \nabla \mathrm{H})\mathbf{n}\, d\omega \quad \text{(surface), from } (2{\cdot}55). \tag{4·3}$$

$$\mathrm{S}r'' = \tfrac{1}{4}\int \mathrm{grad}\mathrm{H}.\ \mathrm{div}(\mathrm{grad}\mathrm{H}.\mathrm{div}\ \mathrm{grad}\mathrm{H})\, d\omega$$

$$- \tfrac{1}{2}\int \mathrm{grad}\ \mathrm{H}\ \mathrm{div}\ \mathrm{grad}\ \nabla \mathrm{H}\, d\omega$$

$$= \tfrac{1}{4}\int (\mathrm{div}\ \mathrm{grad}\mathrm{H}.\ \mathrm{grad}\mathrm{H}\mathrm{grad})(\mathrm{grad}\mathrm{H})\, d\omega + \tfrac{1}{4}\int \mathrm{grad}\mathrm{H}^2 \mathrm{div}\ \mathrm{grad}\mathrm{H}\, d\omega$$

$$+ \tfrac{1}{2}\int (\mathrm{grad}\ \nabla \mathrm{H}\ \mathrm{grad})(\mathrm{grad}\ \mathrm{H})\, d\omega - \tfrac{1}{2}\int \mathrm{grad}\mathrm{H}\,(\mathbf{n}.\ \mathrm{grad}\ \nabla \mathrm{H})\, d\omega,$$

from (2·4),

$$= \tfrac{1}{2}\int (\mathrm{grad}\ \mathrm{H}\ \mathrm{grad})(\mathrm{grad}\ \nabla \mathrm{H})\, d\omega + \tfrac{1}{2}\int (\mathrm{grad}\ \nabla \mathrm{H}\ \mathrm{grad})(\mathrm{grad}\ \mathrm{H})\, d\omega$$

$$+ \tfrac{1}{4}\int \mathrm{grad}\ \mathrm{H}^2.\ \mathrm{div}\ \mathrm{grad}\ \mathrm{H}\, d\omega - \tfrac{1}{2}\int \mathrm{H}\ \mathrm{grad}\ \nabla \mathrm{H}\, d\omega,$$

from (2·54) and ordinary del-calculus,

$$= \tfrac{1}{2}\int \mathrm{grad}\ (\mathrm{grad}\ \mathrm{H}.\mathrm{grad}\ \nabla \mathrm{H})\, d\omega - \tfrac{1}{2}\int \mathrm{H}.\mathrm{grad}\ \nabla \mathrm{H}\, d\omega$$

$$+ \tfrac{1}{4}\int \mathrm{grad}\ \mathrm{H}^2.\mathrm{div}\ \mathrm{grad}\ \mathrm{H}\, d\omega,$$

from ordinary del-calculus,

$$= \tfrac{1}{2}\int \mathbf{n}\,(\mathrm{grad}\ \mathrm{H}.\ \mathrm{grad}\ \nabla \mathrm{H})\, d\omega - \tfrac{3}{2}\int \mathbf{n}.\ \mathrm{H}\nabla \mathrm{H}\, d\omega$$

$$+ \tfrac{1}{2}\int \nabla \mathrm{H}\ \mathrm{grad}\mathrm{H}\, d\omega - \tfrac{1}{2}\int (\mathrm{grad}\ \mathrm{H}.\ \mathrm{grad})(\mathrm{grad}\ \mathrm{H}^2)\, d\omega$$

$$+ \tfrac{3}{2}\int \mathrm{H}^2\ \mathrm{grad}\ \mathrm{H}\, d\omega, \quad \text{from } (2{\cdot}1),$$

$$= \tfrac{1}{2}\int \mathbf{n}.\ (\mathrm{grad}\ \mathrm{H}\ \mathrm{grad}\ \nabla \mathrm{H})\, d\omega - \tfrac{3}{2}\int \mathbf{n}\ \mathrm{H}\nabla \mathrm{H}\, d\omega + \tfrac{3}{2}\int \mathrm{H}^2\ \mathrm{grad}\ \mathrm{H}\, d\omega,$$

from (2·1) and ordinary del-calculus,

$$= \tfrac{1}{2}\int \mathbf{n}\,(\mathrm{grad}\ \mathrm{H}\ \mathrm{grad}\ \nabla \mathrm{H})\, d\omega - \tfrac{3}{2}\int \mathbf{n}\ \mathrm{H}\nabla \mathrm{H}.\, d\omega$$

$$+ \tfrac{3}{2}\int \mathbf{n}\ \mathrm{H}^3\, d\omega \quad \text{(space)} \tag{4·4}$$

$$= \int \{\mathrm{H}^3 - \mathrm{H}\nabla \mathrm{H} + \tfrac{1}{2}\ \mathrm{grad}\ \mathrm{H}\ \mathrm{grad}\ \nabla \mathrm{H}\}\ \mathbf{n}\, d\omega \text{ (surface)}, \tag{4·5}$$

by (2·53) to (2·56),

$$4\mathrm{V}r''' = \tfrac{1}{4}\int \mathrm{grad}\ \mathrm{H}^2\ \mathrm{div}\ (\mathrm{grad}\ \mathrm{H}.\ \mathrm{div}\ \mathrm{grad}\ \mathrm{H})\, d\omega$$

$$-\tfrac{1}{8}\int \mathrm{grad}\ \mathrm{H}^2\ \mathrm{div}\ \mathrm{grad}\ \nabla\ \mathrm{H}\, d\omega$$

$$= \tfrac{1}{4}\int(\text{div grad H grad H grad})(\text{grad H}^2)\,d\omega + \tfrac{1}{8}\int \text{grad H}^3 \text{ div grad H}\,d\omega$$
$$+ \tfrac{1}{8}\int(\text{grad}\,\nabla\text{H grad})(\text{grad H}^2)d\omega, \quad \text{from (2·4)},$$
$$= \tfrac{1}{2}\int \text{H grad}\,\nabla\text{H div grad H}d\omega - \tfrac{1}{2}\int(\text{grad H.}\,\nabla\text{H div grad H})d\omega$$
$$+ \tfrac{1}{8}\int \text{grad H}^3 \text{ div grad H}\,d\omega + \tfrac{1}{8}\int(\text{grad}\,\nabla\text{H grad})(\text{grad H}^2)\,d\omega,$$
from ordinary del-calculus,

$$= \tfrac{1}{4}\int(\text{grad H grad})(\text{H grad}\,\nabla\text{H})\,d\omega - \tfrac{1}{2}\int \text{H}^2 \text{ grad}\,\nabla\text{H}\,d\omega$$
$$+ \tfrac{1}{2}\int(\text{grad H grad})(\text{grad H.}\,\nabla\text{H})d\omega - \int \text{H grad H.}\nabla\text{H}d\omega$$
$$+ \tfrac{1}{8}\int \text{grad H}^3\text{div grad H}\,d\omega + \tfrac{1}{4}\int(\text{H grad}\,\nabla\,\text{H grad})(\text{grad H})d\omega$$
$$+ \tfrac{1}{4}\int \text{grad H (grad H. grad}\,\nabla\text{H})d\omega.$$

Remembering that if **A** and **B** are two vectors, then

$$\text{grad}\,(\mathbf{A}.\,\mathbf{B}) = \mathbf{B}\times\text{rot}\,\mathbf{A} + \mathbf{A}\times\text{rot}\,\mathbf{B} + (\mathbf{B}\,\text{grad})\mathbf{A} + (\mathbf{A}\text{grad})\mathbf{B},$$

and combining suitably, the above expression reduces to

$$\tfrac{1}{4}\int \text{grad (grad H. H grad}\,\nabla\text{H})d\omega - \tfrac{1}{4}\int \text{grad H}\times(\text{grad H}\times\text{grad}\,\nabla\text{H})d\omega$$
$$- \tfrac{1}{2}\int \text{H}^2 \text{ grad}\,\nabla\text{H}d\omega + \tfrac{1}{4}\int \nabla\text{H grad}\,\nabla\text{H}d\omega$$
$$+ \tfrac{1}{2}\int \text{grad H.(grad H grad}\,\nabla\text{H})d\omega - \tfrac{1}{2}\int \text{grad H}^2\,\nabla\text{H}d\omega$$
$$+ \tfrac{1}{8}\int \text{grad H}^3 \text{ div grad H}d\omega + \tfrac{1}{4}\int \text{grad H.(grad H grad}\,\nabla\text{H})d\omega$$
from ordinary del-calculus,

$$= \tfrac{1}{4}\int \text{n (H grad H grad}\,\nabla\text{H})d\omega - \tfrac{1}{4}\int \text{grad H}\times(\text{grad H}\times\text{grad}\,\nabla\text{H})d\omega$$
$$-2\int \text{n. H}^2\,\nabla\,\text{H}d\omega + \tfrac{1}{4}\int \text{n}\,(\nabla\text{H})^2\,d\omega + \tfrac{3}{4}\int \text{grad H(grad H grad}\nabla\text{H})\,d\omega$$
$$+ \tfrac{1}{8}\int \text{grad H}^3 \text{ div grad H}d\omega, \text{ from (2·1) and ordinary del-calculus.}$$

But

$$\tfrac{1}{8}\int \text{grad H}^3\text{div grad H}d\omega$$
$$= -\tfrac{1}{8}\int(\text{grad H grad})(\text{grad H}^3)\,d\omega + \tfrac{3}{8}\int \text{H grad H}^3 d\omega, \text{ from (2·4)},$$
$$= -\tfrac{1}{2}\int \text{H}^2 \text{ grad}\,\nabla\text{H}\,d\omega - \int \text{grad H (grad H grad H}^2)\,d\omega + 6\int \text{n H}^4\,d\omega,$$
from (2·1) and ordinary del-calculus

$$= -2\int \text{n H}^2\,\nabla\text{H}d\omega - \tfrac{1}{2}\int \text{grad H}^2.\,\nabla\text{H}\,d\omega + 6\int \text{n H}^4 d\omega.$$

Also remembering that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{C}\,\mathbf{A}) - \mathbf{C}(\mathbf{AB})$,

we have

$$-\tfrac{1}{4}\int \text{grad H} \times (\text{grad H} \times \text{grad } \nabla\text{H})\, d\omega$$

$$= -\tfrac{1}{4}\int \text{grad H}\,(\text{grad } \nabla\text{H} . \text{grad H})\, d\omega + \tfrac{1}{4}\int \nabla\text{H grad } \nabla\text{H}\; d\omega$$

$$= \tfrac{1}{4}\int \text{grad H}\,(\text{grad } \nabla\text{H} . \text{grad H})\, d\omega + \tfrac{1}{4}\int \mathbf{n}\,(\nabla\text{H})^2\, d\omega, \text{ from } (2{\cdot}1).$$

Therefore

$$\begin{aligned} 4\text{V}r''' &= 6\int \mathbf{n}\,\text{H}^4 d\omega - \tfrac{1}{2}\int \nabla\text{H grad H}^2 d\omega + \tfrac{1}{2}\int \mathbf{n}\,(\nabla\text{H})^2 d\omega \\ &\quad - 4\int \mathbf{n}\,\text{H}^2\,\nabla\text{H}\, d\omega + \tfrac{1}{2}\int \mathbf{n}\,(\text{H grad H grad } \nabla\text{H}) d\omega \\ &\quad + \tfrac{1}{2}\int \text{grad H}\,(\text{grad H} . \text{grad } \nabla\text{H}) d\omega \quad \text{(space)} \qquad (4{\cdot}6) \\ &= \int \{\text{H}^4 - 2\text{H}^2\,\nabla\text{H} + \text{H}\,(\text{grad H} . \text{grad})\,(\nabla\text{H}) + \tfrac{1}{2}(\nabla\text{H})^2\}\,\mathbf{n}\, d\omega \\ &\quad + \tfrac{1}{2}\int \{(\text{grad H grad})\,(\nabla\text{H})\}\,\text{grad H} d\omega \text{ (surface)}, \qquad (4{\cdot}7) \end{aligned}$$

by (2·53) to (2·56).

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8

Equation de Laplace dans l'espace a deux dimensions.

PAR

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Le but du présent article est de donner une esquisse de quelques-uns des résultats touchant à l'équation de Laplace dans l'espace euclidien à deux dimensions. Avec cela nous faisons notre devoir de donner à l'exposé un caractère élémentaire, original et synthétique possible.

Equations aux dérivées partielles du second ordre.

Considérons l'équation du second ordre linéaire en r, s, t, p, q, z,

$$Ar + 2Bs + Ct - F(x, y, z, p, q) = 0, \tag{1}$$

dans laquelle A, B, C désignent des fonctions quelconques de x et y et p, q, r, s, t sont paramètres connus de Monge.

L'équation (1) on peut écrire aussi sous la forme

$$Ar + 2Bs + Ct + Dp + Eq + Fz = G, \tag{1'}$$

où A, B, C, D, E, F, G désignent des fonctions de x et y seuls ; en particulier, si $G = 0$, l'équation (1') on dit linéaire et homogène.

Caractéristiques. La notion de *caractéristique* a une grande importance dans la classification des équations aux dérivées partielles du second ordre. Pour obtenir l'équation de caractéristiques nous voulons employer un tel moyen. Soient

$$Ar + 2Bs + Ct - F(x, y, z, p, q) = 0, \tag{a}$$

$$rdx + sdy - dp = 0, \tag{b}$$

$$sdx + tdy - dq = 0. \tag{c}$$

L'élimination des r et s entre (a), (b) et (c) fournira

$$\left\{ A\left(\frac{dp}{dx} - \frac{dq}{dx} \cdot \frac{dy}{dx}\right) + 2B\,\frac{dq}{dx} - F \right\} + \left\{ A\left(\frac{dy}{dx}\right)^2 - 2B\,\frac{dy}{dx} + C \right\} t = 0. \tag{d}$$

On satisfera à cette équation en prenant

$$A\left(\frac{dy}{dx}\right)^2 - 2B\,\frac{dy}{dx} + C = 0, \tag{2}$$

alors

$$A\left(\frac{dp}{dx} - \frac{dq}{dx}\,\frac{dy}{dx}\right) + 2B\,\frac{dq}{dx} - F = 0. \tag{3}$$

Ces équations on peut écrire et d'une telle manière

$$A dy^2 - 2B dx dy + C dx^2 = 0, \tag{2'}$$

et

$$A(dq dy - dp dx) - 2B dq dx + F dx^2 = 0, \tag{3'}$$

ou sous la forme des déterminants

$$\begin{vmatrix} A & 2B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0, \tag{2''}$$

et

$$\begin{vmatrix} A & 2B & F \\ dx & dy & dp \\ 0 & dx & dq \end{vmatrix} = 0. \tag{3''}$$

Pour notre but il suffit de considérer l'équation (2).

L'équation différentielle (2 ou 2') du premier ordre et du second degré on appelle l'équation différentielle des caractéristiques. Cette équation nous pouvons décomposer en deux équations du premier ordre et du premier degré, à savoir

$$A\,dy - (B + \sqrt{B^2 - AC})dx = 0, \tag{4}$$

et

$$A\,dy - (B - \sqrt{B^2 - AC})dx = 0. \tag{4'}$$

Les deux dernières équations et nous donnent les deux familles de *courbes caractéristiques* par l'intégration de ces équations. Soient

$$\xi\,(x, y) = \text{const.}, \quad \eta\,(x, y) = \text{const.}, \tag{e}$$

les deux intégrales ainsi obtenues. En ayant égard à la relation (2'), il est aisé de voir que ξ et η satisfont à l'équation du premier ordre

$$A\left(\frac{\partial f}{\partial x}\right)^2 + 2\,B\,\frac{\partial f}{\partial x}\,\frac{\partial f}{\partial y} + C\left(\frac{\partial f}{\partial y}\right)^2 = 0. \tag{5}$$

Maintenant nous constituerons la classification suivante des équations différentielles :

1) Si $B^2-AC>0$, l'équation (1) est du type *hyperbolique*;

2) Si $B^2-AC=0$, l'équation (1) est du type *parabolique*;

3) Si $B^2-AC<0$, l'équation (1) est du type *elliptique*.

Dans le premier cas, les deux familles de caractéristiques sont réelles et distinctes; dans le deuxième cas, les caractéristiques sont réelles et confondues et dans le dernier cas les caractéristiques sont imaginaires.

Et finalement il faut prendre ξ et η pour nouvelles variables si l'on veut ramener l'équation (1) aux *formes canoniques*. En résultat de nos transformations l'équation proposée se change en une nouvelle équation de même forme

$$A_1\frac{\partial^2 z}{\partial \xi^2}+2B_1\frac{\partial^2 z}{\partial \xi\partial \eta}+C_1\frac{\partial^2 z}{\partial \eta^2}-F_1\left(\xi,\ \eta,\ z,\ \frac{\partial z}{\partial \xi},\frac{\partial z}{\partial \eta}\right)=0, \quad (6)$$

$$\left.\begin{aligned} \text{où } A_1 &= A\left(\frac{\partial \xi}{\partial x}\right)^2+2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right)+C\left(\frac{\partial \xi}{\partial y}\right)^2,\\ B_1 &= A\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}+B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}+\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)+C\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y},\\ C_1 &= A\left(\frac{\partial \eta}{\partial x}\right)^2+2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right)+C\left(\frac{\partial \eta}{\partial y}\right)^2. \end{aligned}\right\} \quad (7)$$

Équation du type elliptique. Il est intéressant pour nous de considérer seulement le cas troisième. En ce cas nous avons les caractéristiques imaginaires. Prenons $\xi+i\eta=X(x,y)$, $\xi-i\eta=Y(x,y)$. Si dans la relation (5) on pose $X=f=\xi+i\eta$, on trouvera

$$A_1-C_1+2B_1 i=0, \quad (g)$$

Donc $\qquad A_1=C_1, \quad (8) \qquad$ et $\qquad B_1=0. \quad (9)$

En ayant égard à ces formules, on ramène l'équation (6) à la forme canonique

$$\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2}+f\left(\xi,\ \eta,\ z,\ \frac{\partial z}{\partial \xi},\ \frac{\partial z}{\partial \eta}\right)=0, \quad (10)$$

où f est la fonction linéaire par rapport aux z, $\frac{\partial z}{\partial \xi}$ et $\frac{\partial z}{\partial \eta}$. On peut, si l'on veut, représenter l'équation (10) sous la forme suivante

$$\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2}+a\frac{\partial z}{\partial \xi}+b\frac{\partial z}{\partial \eta}+cz+g=0. \tag{10'}$$

Si l'on pose dans cette équation $a=b=c=g=0$, on obtiendra

$$\Delta z=\nabla^2 z=\frac{\partial^2 z}{\partial \xi^2}+\frac{\partial^2 z}{\partial \eta^2}=0. \tag{11}$$

L'équation (11) connue sous le nom de l'*équation de Laplace*. Cette équation a une grande importance en Analyse et en Physique mathématique. Comme on voit, elle est le cas particulier des équations du type elliptique (E. Picard, Leçons sur quelqu. typ. simpl. d'équat. aux dériv. part., 1927. E. Goursat. Cours d'Anal. mathém., t. III, 1927., L. Bieberbach, Differential-gleichungen, 1926).

Conditions de Cauchy-Riemann et équation de Laplace.

Soit $w=f(z)$ une fonction continue uniforme de la variable complexe z définie dans un domaine D. On peut écrire

$$w=f(z)=f(x+yi)=P+Qi, \tag{h}$$

où P et Q sont des fonctions réelles continues de deux variables réelles x et y. Cherchons dans quels cas la dérivée $\frac{dw}{dz}$ est complétement determinée et ne dépend que du point quelconque M.

On a

$$dw=\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy+i\left(\frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy\right)$$
$$=(r\cos\zeta+i\operatorname{sn}\zeta), \tag{i}$$

d'où $$\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy=r\cos\zeta,\qquad \frac{\partial Q}{\partial x}dx+\frac{\partial Q}{\partial y}dy=r\operatorname{sn}\zeta. \tag{j}$$

En outre

$$dz=dx+idy=\rho(\cos\psi+i\operatorname{sn}\psi), \tag{k}$$

d'où $$dx=\rho\cos\psi,\quad dy=\rho\operatorname{sn}\psi. \tag{l}$$

Soit maintenant $$\zeta=\psi+\theta. \tag{m}$$

Alors

$$\left.\begin{aligned}\frac{\partial P}{\partial x}\,\rho \mathrm{cs}\,\psi + \frac{\partial P}{\partial y}\,\rho \mathrm{sn}\,\psi &= r\mathrm{cs}(\psi+\theta)\\ &= r\mathrm{cs}\psi\mathrm{cs}\,\theta - r\mathrm{sn}\psi\mathrm{sn}\theta,\\ \frac{\partial Q}{\partial x}\,\rho \mathrm{cs}\,\psi + \frac{\partial Q}{\partial y}\,\rho \mathrm{sn}\,\psi &= r\mathrm{sn}(\psi+\theta)\\ &= r\mathrm{sn}\psi\mathrm{cs}\theta + r\mathrm{cs}\psi\mathrm{sn}\theta.\end{aligned}\right\} \qquad (n)$$

De là on peut conclu e que

$$\left.\begin{aligned}\frac{\partial P}{\partial x} &= \frac{r}{\rho}\,\mathrm{cs}\theta, & -\frac{\partial P}{\partial y} &= \frac{r}{\rho}\mathrm{sn}\theta,\\ \frac{\partial Q}{\partial x} &= \frac{r}{\rho}\,\mathrm{sn}\theta, & \frac{\partial Q}{\partial y} &= \frac{r}{\rho}\mathrm{cn}\theta.\end{aligned}\right\} \qquad (12)$$

Ces dernières formules nous donnent enfin

$$\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}, \qquad \frac{\partial P}{\partial y} = -\frac{\partial Q}{\partial x}. \qquad (13)$$

Les relations (13) et l'on se nomme *conditions de Cauchy-Riemann.*

Les formules (12) donnent de même

$$\left.\begin{aligned}&\frac{\partial P}{\partial x}\cdot\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\cdot\frac{\partial Q}{\partial y} = 0, \quad \frac{\partial P}{\partial x}\cdot\frac{\partial Q}{\partial x} + \frac{\partial P}{\partial y}\cdot\frac{\partial Q}{\partial y} = 0\,;\\ &\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial P}{\partial y}\right)^2 = \left(\frac{\partial Q}{\partial x}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2,\\ &\left(\frac{\partial P}{\partial x}\right)^2 + \left(\frac{\partial Q}{\partial x}\right)^2 = \left(\frac{\partial P}{\partial y}\right)^2 + \left(\frac{\partial Q}{\partial y}\right)^2.\end{aligned}\right\} \qquad (14)$$

On sait que pour tout système de fonctions P et Q verifiant les conditions de C.-R., la continuité des derivées du premier ordre entraîne l'existence et la continuité des dérivées d'un ordre quelconque.

Maintenant en différentiant la première des relations (13) par rapport à x, la seconde par rapport à y et en ajoutant les deux relations obtenues, on aura

$$\Delta P = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0. \qquad (15)$$

On trouve de même

$$\Delta Q = \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} = 0. \qquad (16)$$

Les fonctions P et Q satisfont ainsi à l'équation de la forme

$$\Delta U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{17}$$

On appelle fonction U (x, y), qui satisfait à l'équation (17), fonction *harmonique* ou aussi *fondamentale* (H. Bouasse, Cours de mathématiques générales, 1927).

Conditions de Beltrami-Bernstein et équation de Laplace sur la surface.

Prenons les relations (8) et (9), c'est-à-dire

$$A\left(\frac{\partial \xi}{\partial x}\right)^2 + 2B\left(\frac{\partial \xi}{\partial x}\right)\left(\frac{\partial \xi}{\partial y}\right) + C\left(\frac{\partial \xi}{\partial y}\right)^2$$

$$= A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\left(\frac{\partial \eta}{\partial x}\right)\left(\frac{\partial \eta}{\partial y}\right) + C\left(\frac{\partial \eta}{\partial y}\right)^2, \tag{18}$$

$$A\,\frac{\partial \xi}{\partial x}\,\frac{\partial \eta}{\partial x} + B\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial x}\right) + C\,\frac{\partial \xi}{\partial y}\,\frac{\partial \eta}{\partial y} = 0. \tag{19}$$

Les conditions (18) et (19) donnent la possibilité de ramener l'équation (1) à la forme canonique (10), si l'on introduit deux nouvelles variables ξ et η à la place de x et y dans l'équation proposée. Cette canonisation est ainsi équivalente à la représentation conforme d'une surface S sur un plan. L'expression du carré de l'élément d'arc sur cette surface sera

$$ds^2 = C dx^2 - 2B dx dy + A dy^2. \tag{20}$$

En résolvant les équations (18) et (19) par rapport à $\frac{\partial \xi}{\partial x}$ et $\frac{\partial \xi}{\partial y}$, on tire successivement

$$\frac{\partial \xi}{\partial x}\left(A\,\frac{\partial \eta}{\partial x} + B\,\frac{\partial \eta}{\partial y}\right) = -\frac{\partial \xi}{\partial y}\left(B\,\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial x}\right), \tag{a}$$

$$\left(\frac{\partial \xi}{\partial y}\right)^2 \left\{ A\left(B\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial y}\right)^2 \right.$$

$$-2B\left(B\frac{\partial \eta}{\partial x} + C\frac{\partial \eta}{\partial y}\right)\left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)$$

$$\left. + C\left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)^2 \right\}$$

$$= \left\{ A\left(\frac{\partial \eta}{\partial x}\right)^2 + 2B\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + C\left(\frac{\partial \eta}{\partial y}\right)^2 \right\}$$

$$\times\left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)^2. \qquad (p)$$

En simplifiant cette égalité, il vient

$$\left(\frac{\partial \xi}{\partial y}\right)^2 (AC - B^2) = \left(A\frac{\partial \eta}{\partial x} + B\frac{\partial \eta}{\partial y}\right)^2, \qquad (q)$$

d'où

$$\frac{\partial \xi}{\partial y} = \pm \frac{A\dfrac{\partial \eta}{\partial x} + B\dfrac{\partial \eta}{\partial y}}{\sqrt{AC - B^2}}. \qquad (21)$$

Et par suite

$$\frac{\partial \xi}{\partial x} = \mp \frac{B\dfrac{\partial \eta}{\partial x} + C\dfrac{\partial \eta}{\partial y}}{\sqrt{AC - B^2}} \qquad (22)$$

Les relations (21) et (22) on peut appeller *conditions de Beltrami-Bernstein.* Ces conditions sont la généralisation du système (18). L'expression $\xi + i\eta$ peut être considérée comme une fonction du point (x, y) sur la surface S.

La condition d'intégrabilité de la différentielle totale de la fonction ξ de deux variables x et y, à savoir $\dfrac{\partial^2 \xi}{\partial x \partial y} = \dfrac{\partial^2 \xi}{\partial y \partial x}$, nous donnera l'équation cherchée

$$\frac{\partial}{\partial x}\left(\frac{A\dfrac{\partial \eta}{\partial x} + B\dfrac{\partial \eta}{\partial y}}{\sqrt{AC - B^2}}\right) + \frac{\partial}{\partial y}\left(\frac{B\dfrac{\partial \eta}{\partial x} + C\dfrac{\partial \eta}{\partial y}}{\sqrt{AC - B^2}}\right) = 0. \qquad (23)$$

L'équation (23) est sur la surface S l'analogue de l'équation de Laplace sur le plan. Cette équation on peut appeler *l'équation de Beltrami-Bernstein.* Analoguement nous aurions écrit l'équation seconde de Beltrami-Bernstein pour la fonction ξ.

Il est facile de voir que l'équation de Bernstein et celle de Beltrami sont en essence identiques. A cet effet il ne suffit que decomparer ces équations et les deux formes quadratiques binaires.

Il est intéressant aussi de remarquer que les relations (14) sont les cas particuliers des relations (19) (E Picard, Traité d'Analyse, t. II., p. 8-9, 1905. S. Bernstein, La Thèse, en russe, L'étude et l'intégration des équations aux dérivées partielles du type elliptique, pp. 113-115, 1908).

Equation tensorielle de Laplace.

Définitions. Soit un vecteur $\overline{ds}$ dans l'espace euclidien à n dimensions a la forme suivante

$$\overline{ds} = \sum_i e_i \, dx^i = \sum_i e^i \, dx_i, \tag{24}$$

où e_i et e^i sont respectivement n *vecteurs fondamentaux* linéairement indépendants dans chaque système.

En prenant maintenant le produit scalaire d'un vecteur $\overline{ds}$ par lui-même on obtient

$$ds^2 = \sum_{ik} g_{ik} \, dx^i dx^k = \sum_{ik} g^{ik} dx_i dx_k, \tag{25}$$

où

$$g_{ik} = g_{ki} = e_i e_k, \tag{26}$$

et

$$g^{ik} = g^{ki} = e^i e^k. \tag{27}$$

De plus, on a

$$e^i e_k = \delta^i_k, \quad (28), \qquad \sum_k g_{ik} g^{kl} = \delta^l_i, \tag{29}$$

où le symbole

$$\delta^s_r = \begin{cases} 1 \text{ pour } r = s, \\ 0 \qquad\quad r \neq s. \end{cases}$$

Aux grandeurs g_{ik} et g^{kl} on donne parfois le nom de tenseurs fondamentaux: g_{ik} est le tenseur *covariant* et g^{kl} est le tenseur *contrevariant.* Les grandeurs g_i et g^i sont les composants de deux tenseurs symétriques T_g et $(T_g)^{-1}$, à savoir

$$T_g = \|g_{ik}\|, \quad (30), \qquad (T_g)^{-1} = \|g^{ik}\|. \tag{31}$$

Dans ces formules au lieu de symbole ‖ ‖ pour la signification de la matrice on peut aussi employer le symbole ( ).

En appelant g le déterminant des g_{ik} et G^{ik} le mineur de g relatif au terme g_{ik}, on obtiendra les relations suivantes

$$g = |g_{ik}| \qquad (32), \qquad \frac{1}{g} = |g^{ik}|, \tag{33}$$

$$g^{ik} = \frac{G^{ik}}{g} = \frac{1}{g}\frac{\partial g}{\partial g_{ik}}\ ; \tag{34}$$

ici

$$g = v^2 = |e_i|^2 \tag{35}$$

(v est le volume du parallélépipède coordonné).

Enfin on écrira encore une formule, savoir

$$\nabla^2 = g^{ik}\frac{\partial^2}{\partial x_i \partial x_k}, \tag{36}$$

où

$$\nabla = \sum_i e^i \frac{\partial}{\partial x_i}. \tag{37}$$

Si les axes coordonnées sont rectangulaires, le Laplacien est

$$\nabla^2 = \sum_i \frac{\partial^2}{\partial x_i^2}, \tag{38}$$

car

$$g^{ik} = \begin{cases} 1 & \text{pour } i=k, \\ 0 & \phantom{\text{pour }} i \neq k. \end{cases}$$

Extrémés des intégrales doubles.

On cherche le maximum et le minimum de l'intégrale double

$$\mathrm{I} = \iint_{\mathrm{D}} \mathrm{F}(x, y, u, u_x, u_y)\, dxdy, \tag{38'}$$

étendue à un domaine déterminé D. Dans la formule (38), F est une fonction donnée, continue ainsi que ses derivées partielles des divers ordres et u est une fonction de deux variables indépendantes x, y. Enfin les grandeurs u_x et u_y sont respectivement les dérivées prises par rapport à x et y. Soit encore $\delta u = 0$ sur le bord du domaine D. En vertu de nos conditions, nous aurons

$$\delta\,\mathrm{I} = \iint_{\mathrm{D}} \left[\frac{\partial \mathrm{F}}{\partial u}\delta u + \frac{\partial \mathrm{F}}{\partial u_x}\delta u_x + \frac{\partial \mathrm{F}}{\partial u_y}\delta u_y\right] dxdy, \tag{39}$$

ou aussi

$$\delta \mathrm{I} = \iint_{\mathrm{D}} \left[\mathrm{U}\delta u + \mathrm{X}\frac{\partial\, \delta u}{\partial x} + \mathrm{Y}\frac{\partial\, \delta u}{\partial y} \right] dxdy, \tag{40}$$

en posant, pour abréger,

$$\frac{\partial \mathrm{F}}{\partial u} = \mathrm{U}, \quad \frac{\partial \mathrm{F}}{\partial u_x} = \mathrm{X}, \quad \frac{\partial \mathrm{F}}{\partial u_y} = \mathrm{Y}.$$

Intégrant par parties sous le signe d'intégration et observant que δu s'annule par hypothèse sur le bord du domaine D, on obtiendra

$$\delta \mathrm{I} = \iint_{\mathrm{D}} \left[\mathrm{U} - \frac{\partial \mathrm{X}}{\partial x} - \frac{\partial \mathrm{Y}}{\partial y} \right] dxdy. \tag{41}$$

Comme la variation δu est arbitraire dans tout le domaine d'intégration, on trouvera de la relation (41) l'équation d'Euler, savoir

$$\mathrm{U} - \frac{\partial \mathrm{X}}{\partial x} - \frac{\partial \mathrm{Y}}{\partial y} = 0. \tag{42}$$

L'équation (42) on peut écrire d'une telle façon

$$-[\mathrm{F}]_u = \frac{\partial}{\partial x}\, \mathrm{F}_{u_x} + \frac{\partial}{\partial y}\, \mathrm{F}_{u_y} - \mathrm{F}_u = 0. \tag{43}$$

L'expression $[\mathrm{F}]_u$ on peut appeler, selon Hilbert, " la dérivée de la variation " (variationsableitung), savoir de F par rapport à u. Cette dérivée joue le même rôle dans le calcul des variations que la dérivée ordinaire dans la théorie des maxima et des minima.

Transformation de variables. Si l'on pose maintenant

$$x = x(\xi, \eta), \quad y = y(\xi, \eta), \tag{r}$$

on trouvera

$$\mathrm{F}(x, y, u, u_x, u_y) = \Phi\,(\xi, \eta, u, u_\xi, u_\eta), \tag{s}$$

$$\iint_{\mathrm{D}} \mathrm{F}\, dxdy = \iint_{\mathrm{D}_1} \Phi\, \frac{\partial(x, y)}{\partial(\xi, \eta)}\, d\xi d\eta, \tag{t}$$

$$\iint_{\mathrm{D}} [\mathrm{F}]_u\, \delta u dxdy = \iint_{\mathrm{D}_1} \left[\Phi\, \frac{\partial(x, y)}{\partial(\xi, \eta)} \right]_u \delta u\, d\xi d\eta\,; \tag{44}$$

d'où il suit

$$[\mathrm{F}]_u = \frac{\partial(\xi, \eta)}{\partial(x, y)} \left[\Phi \frac{\partial(x, y)}{\partial(\xi, \eta)} \right]_u, \tag{45}$$

car

$$d\xi d\eta = \frac{\partial(\xi, \eta)}{\partial(x, y)} \; dx dy,$$

et de plus on suppose que le champ d'intégration soit invariable, *c.-à.-d.* $\mathrm{D}_1 = \mathrm{D}$.

La formule (45) joue un rôle important dans la question de la transformation de variables.

Transformation de l'expression Δu. Soit

$$\mathrm{F} = \tfrac{1}{2}(u^2_x + u^2_y), \tag{u}$$

et

$$x = x(\xi_1, \xi_2), \quad y = y(\xi_1, \xi_2). \tag{u'}$$

On a

$$ds^2 = dx^2 + dy^2 = \sum_{ik} g_{ik} d\xi_i d\xi_k , \tag{v}$$

où

$$g_{ik} = \frac{\partial x}{\partial \xi_i} \frac{\partial x}{\partial \xi_k} + \frac{\partial y}{\partial \xi_i} \frac{\partial y}{\partial \xi_k} . \tag{w}$$

Il est aisé de voir que

$$g = \begin{vmatrix} \dfrac{\partial x}{\partial \xi_1} & \dfrac{\partial x}{\partial \xi_2} \\[2ex] \dfrac{\partial y}{\partial \xi_1} & \dfrac{\partial y}{\partial \xi_2} \end{vmatrix}^2 = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}. \tag{x}$$

Il n'est pas difficile de calculer aussi l'expression $u^2_x + u^2_y$. En effet

$$u^2_x + u^2_y = \sum_{ik} g^{ik} u_i u_k, \left(u_i = \frac{\partial u}{\partial \xi_i} \right), \tag{y}$$

où

$$g^{ik} = \frac{\partial \xi_i}{\partial x} \cdot \frac{\partial \xi_k}{\partial x} + \frac{\partial \xi_i}{\partial y} \cdot \frac{\partial \xi_u}{\partial y} , \tag{z}$$

et

$$\sum_k g_{ik} g^{kl} = \delta_i^l ; \tag{z'}$$

ici le symbole δ_i^l a le sens précédent.

Or $$\frac{\partial(x,y)}{\partial(\xi_1,\xi_2)} = \sqrt{g}\ , \tag{A}$$

et $$[\mathrm{F}]_u = -\Delta u = -(u_{xx}+u_{yy}). \tag{B}$$

Par conséquent la relation (45) nous donnera

$$-\Delta u = \frac{1}{2\sqrt{g}}\left[\sqrt{g}\ \sum_{ik} g^{ik}\ u_i u_k\right]_u. \tag{46}$$

On en conclura

$$-\Delta u = \frac{1}{2\sqrt{g}}\left[\sqrt{g}\ (g^{11}\ u_1^2+2g^{12}u_1u_2+g^{22}u_2^2)\right]_u ,$$

$$\Delta u = \frac{1}{2\sqrt{g}}\frac{\partial}{\partial\xi_1}\left[\sqrt{g}\ (2g^{11}\ u_1+2g^{12}u_2)\right]$$

$$+\frac{1}{2\sqrt{g}}\frac{\partial}{\partial\xi_2}\left[\sqrt{g}\ \ (2g^{21}u_1+2g^{22}u_2)\right] ,$$

$$\Delta u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_1}\left[\sqrt{g}\ \sum_k g^{1k}u_k\right]+\frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_2}\left[\sqrt{g}\ \sum_k g^{2k}u_k\right],$$

$$\Delta u = \frac{1}{\sqrt{g}}\sum_i\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\ \sum_k g^{ik}\ u_k\right]. \tag{47}$$

Cette dernière formule peut s'écrire sommairement d'une telle manière

$$\Delta u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\,[\sqrt{g}\ g^{ik}u_k]. \tag{48}$$

Tel est l'opérateur de Laplace cherché.

Par suite l'équation de Laplace tensorielle a la forme suivante

$$\Delta u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}[\sqrt{g}\ g^{ik}u_k] = 0. \tag{49}$$

Exemples. 1. *Coordonées cartésiennes.* On a

$$ds^2 = dx^2+dy^2, \tag{50}$$

c'est-à-dire,

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{12} = 0, \quad g^{11} = 1,$$
$$g^{22} = 1, \quad g^{12} = 0, \quad g = 1.$$

Par suite, l'équation (49) prendra la forme

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{51}$$

Soient maintenant

$$x = \alpha_1 x_1 + \beta_1 x_2, \quad y = \alpha_2 x_1 + \beta_2 x_2,$$

les formules de transformation. Soient en outre dans ces formules

$$a\cos\alpha = \alpha_1, \quad a\operatorname{sn}\alpha = \alpha_2, \quad b\cos(\alpha+\zeta) = \beta_1, \quad b\operatorname{sn}(\alpha+\zeta) = \beta_2.$$

Et puis $\qquad \operatorname{tg}\alpha = \dfrac{\alpha_2}{\alpha_1}, \quad \operatorname{tg}(\alpha+\zeta) = \dfrac{\beta_2}{\beta_1}.$

Il est facile de voir que

$$\operatorname{tg}\zeta = \frac{\alpha_1\beta_2 - \alpha_2\beta_1}{\alpha_1\beta_1 + \alpha_2\beta_2}, \tag{C}$$

et

$$\cos\zeta = \pm \frac{\alpha_1\beta_1 + \alpha_2\beta_2}{\sqrt{\alpha_1^2+\alpha_2^2}\cdot\sqrt{\beta_1^2+\beta_2^2}}. \tag{C'}$$

Si l'on pose, pour abréger,

$$c^2 = \alpha_1\beta_1 + \alpha_2\beta_2, \tag{D}$$

on obtiendra

$$ds^2 = a^2 dx_1^2 + b^2 dx_2^2 + 2c^2 dx_1 dx_2. \tag{52}$$

En vertu de la formule (C′) nous avons aussi

$$c^2 = ab\cos\zeta. \tag{E}$$

Les tenseurs fondamentaux en notre cas seront telles

$$g_{11} = a^2, \quad g_{22} = b^2, \quad g_{12} = g_{21} = c^2 = ab\cos\zeta\,;$$
$$g^{11} = \frac{b^2}{g} = \frac{1}{a^2\operatorname{sn}^2\zeta}, \quad g^{22} = \frac{a^2}{g} = \frac{1}{b^2\operatorname{sn}^2\zeta},$$
$$g^{12} = g^{21} = \frac{c^2}{g} = -\frac{1}{ab}\cdot\frac{\cos\zeta}{\operatorname{sn}^2\zeta},$$

où $\qquad g = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = a^2b^2 - a^2b^2\cos^2\zeta = a^2b^2\operatorname{sn}^2\zeta.$

Par conséquent la relation (49) prendra la forme

$$\Delta u = \frac{1}{g}\left[b^2\frac{\partial^2 u}{\partial x_1^2} - 2c^2\frac{\partial^2 u}{\partial x_1 \partial x_2} + a^2\frac{\partial^2 u}{\partial x_2^2}\right] = 0, \tag{53}$$

ou finalement

$$\Delta u = \frac{1}{\operatorname{sn}^2\zeta}\left[\frac{1}{a^2}\frac{\partial^2 u}{\partial x_1^2} - \frac{2\cos\zeta}{ab}\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{1}{b^2}\frac{\partial^2 u}{\partial x_2^2}\right] = 0. \tag{54}$$

2. *Coordonées polaires.* On sait que

$$ds^2 = dr^2 + r^2 d\zeta^2. \tag{F}$$

On a successivement

$$\begin{gathered} g_{11} = 1, \quad g_{22} = r^2, \quad g_{12} = 0, \quad g = r^2 ; \\ g^{11} = \frac{g_{22}}{g} = 1, \quad g^{22} = \frac{g_{11}}{g} = \frac{1}{r^2}, \quad g^{12} = 0. \end{gathered} \tag{G}$$

D'après formule (49) nous pouvons écrire

$$\begin{aligned} \Delta u = & \frac{1}{\sqrt{g}}\frac{\partial}{\partial r}\left[\sqrt{g}\left(g^{11}\frac{\partial u}{\partial r} + g^{12}\frac{\partial u}{\partial \zeta}\right)\right] \\ & + \frac{1}{\sqrt{g}}\frac{\partial}{\partial \zeta}\left[\sqrt{g}\left(g^{21}\frac{\partial u}{\partial r} + g^{22}\frac{\partial u}{\partial \zeta}\right)\right] = 0. \end{aligned} \tag{55}$$

En vertu des relations (G) nous aurons

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \zeta}\left(\frac{1}{r}\frac{\partial u}{\partial \zeta}\right) = 0, \tag{56}$$

ou aussi $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \zeta^2} = 0. \tag{57}$$

Telle est l'équation de Laplace en coordonnées polaires.

La résolution de l'équation de Laplace.

Considérons l'équation (15),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Soient $x_1 = mx + ny, \quad y_1 = m_1 x + n_1 y.$

Il est facile d'obtenir

$$\frac{\partial^2 u}{\partial x^2} = \left(m\frac{\partial}{\partial x_1}+m_1\frac{\partial}{\partial y_1}\right)^2 u, \quad \frac{\partial^2 u}{\partial y^2} = \left(n\frac{\partial}{\partial x_1}+n_1\frac{\partial}{\partial y_1}\right)^2 u. \tag{H}$$

De là on voit que la nouvelle équation conserverait la même forme que la première, si l'on posait

$$m^2+n^2 = m_1^2+n_1^2 = 1, \quad mm_1+nn_1 = 0. \tag{I}$$

Or on peut aussi poser

$$m^2+n^2 = m_1^2+n_1^2 = 0, \tag{J}$$

en prenant, par exemple, $m = m_1 = 1, \ n = n_1 = -i.$ (J′)

En ce cas l'équation de Laplace transformée prendra la forme

$$\frac{\partial^2 u}{\partial x_1 \partial y_1} = 0. \tag{58}$$

On en conclut $\dfrac{\partial^2 u}{\partial x_1 \partial y_1} = \dfrac{\partial}{\partial y_1}\left(\dfrac{\partial u}{\partial x_1}\right) = 0.$

Par suite $\dfrac{\partial u}{\partial x_1} = f(x_1),$

$$u = \int f(x_1)dx_1 + \mathrm{F}_2(y_1).$$

Et enfin

$$u = \mathrm{F}_1(x_1)+\mathrm{F}_2(y_1) = \mathrm{F}_1(x+iy)+\mathrm{F}_2(x-iy), \tag{59}$$

La méthode symbolique. On a l'équation

$$\frac{\partial^2 z}{\partial x^2} + \mathrm{A}_1\frac{\partial^2 z}{\partial x \partial y} + \mathrm{A}_2\frac{\partial^2 z}{\partial y^2} = 0, \tag{K}$$

dans laquelle nous supposons A_1, A_2 constants.

Cette équation, comme on sait, on peut représenter symboliquement, à savoir

$$(\mathrm{D}^2+\mathrm{A}_1\mathrm{DD}_1+\mathrm{A}_2\mathrm{D}_1^2)\, z = 0. \tag{K′}$$

En décomposant la première partie de l'égalité (K′), on obtiendra

$$(\mathrm{D}-m_1\mathrm{D}')\,(\mathrm{D}-m_2\mathrm{D}')\, z = 0, \tag{K″}$$

où m_1 et m_2 sont les racines de l'équation caractéristique,

$$m^2+\mathrm{A}_1 m+\mathrm{A}_2 = 0. \tag{L}$$

On a, par exemple,

$$(D-m_k D')z = 0, \quad (k=1,2). \tag{M}$$

En posant, pour abréger, $m_k D' = \lambda$, on peut écrire

$$(D-\lambda)z = 0, \tag{M'}$$

d'où nous trouvons

$$z = e^{\lambda x} C. \tag{N}$$

Soit maintenant $C = \zeta_k(y)$, alors

$$z = e^{\lambda x}\zeta_k(y) = e^{m_k x \frac{\partial}{\partial y}}\zeta_k(y) = \zeta_k(y+m_k x). \tag{N'}$$

En effet

$$f(x+h) = \left(1 + hD_x + \frac{h^2}{1.2}D_x^2 + \ldots\right) f(x) = e^{hD_x} f(x). \tag{O}$$

Il est facile de voir que

$$f(y+hx) = e^{hx\,D_y} f(y). \tag{P}$$

Ainsi l'intégrale générale de (K) prend la forme

$$z = \zeta_1(y+m_1 x) + \zeta_2(y+m_2 x). \tag{Q}$$

Prenons de nouveau l'équation

$$(D^2 + D'^2)\, z = 0. \tag{60}$$

On a $\qquad (D'+iD)(D'-iD)\,z = 0.$

Donc, par exemple, $(D'+iD)z=0$, ou $q+ip=0$; les équations canoniques seront

$$\frac{dx}{i} = \frac{dy}{1} = \frac{dz}{0},$$

d'où on trouve $\quad z = C_1, \quad x - yi = C_2,$

c'est-à-dire, $\quad z = f_1(x-yi).$

Par suite l'intégrale générale de (60) prendra la forme

$$z = f_1(x-yi) + f_2(x+yi).$$

Considérons enfin le cas des racines égales. On a l'équation symbolique

$$(D+mD')^2 z = 0. \tag{61}$$

Posons $$(D+mD')z = u. \tag{R}$$

Donc $$(D+mD')u = 0. \tag{S}$$

Il suit de là $$u = f(y-mx). \tag{T}$$

Par conséquent. $$(D+mD')z = f(y-mx), \tag{T'}$$

ou $$p+mq = f(y-mx). \tag{T''}$$

Le système des équations canoniques est

$$\frac{dx}{1} = \frac{dy}{m} = \frac{dz}{f(y-mx)}, \tag{U}$$

d'où on trouve

$$y-mx = C_1, \quad z-xf(y-mx) = C_2.$$

Par suite l'intégrale générale de l'équation (61) sera

$$z = f_1(y-mx)+xf(y-mx). \tag{X}$$

(R. Courant u. D. Hilbert, Method. der mathem. Physik. I, 1924. E. Madelung, Die mathem. Hilfsmittel des Physik., 1925. A. Forsyth, Lehrbuch der Differentialgleich., 1889. H. Piaggio, An elementary treatise on different. equat., trad., en russe, 1933. H. Malet, Exposé élément. du calcul vectoriel, 1927).

L'équation biharmonique.

En conclusion du présent article considérons encore une équation qui joue un rôle important dans la théorie d'élasticité. Cette équation a la forme suivante

$$\Delta\Delta u = \frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial y^4} = 0. \tag{62}$$

L'équation dernière on peut écrire encore symboliquement d'une telle manière

$$(D^4+2D^2D'^2+D'^4)\,u = (D^2+D'^2)^2 u = 0. \tag{63}$$

L'équation (62) ou (63) on appelle ordinairement *l'équation biharmonique* et son résolution on se dit la *fonction biharmonique*. Il est aisé de résoudre l'équation proposée. En effet l'équation caractéristique $(m^2+1)^2 = 0$ a les racines $m_1 = m_2 = i$, $m_3 = m_4 = -i$. Par conséquent l'intégrale générale de notre équation a la forme suivante

$$u = f_1(x+yi) + y\, f_2(x+yi) + f_3(x-yi) + y f_4(x-yi). \tag{64}$$

La notion *harmonique* ou *fondamentale* on peut généraliser encore plus loin.

En effet on peut former les équations suivantes

$$(D^2 + D'^2)^3 u = 0, \quad (D^2 + D'^2)^4 u = 0, \text{ etc.} \tag{65}$$

Ces équations on peut résoudre au moyen de notre méthode symbolique.

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On Some Simple Distributions Of Stress In Three Dimensions

BY

S. Ghosh

1. The equations of equilibrium of an isotropic elastic solid under no body forces, when referred to polar co-ordinates, are*

$$\left.\begin{aligned}
(\lambda+2\mu)\sin\theta\frac{\partial\Delta}{\partial\theta}-2\mu\left\{\frac{\partial\omega_r}{\partial\phi}-\frac{\partial}{\partial r}(r\omega_\phi\sin\theta)\right\}&=0,\\
(\lambda+2\mu)\frac{1}{\sin\theta}\frac{\partial\Delta}{\partial\phi}-2\mu\left\{\frac{\partial}{\partial r}(r\omega_\theta)-\frac{\partial\omega_r}{\partial\theta}\right\}&=0,\\
(\lambda+2\mu)\,r\sin\theta\frac{\partial\Delta}{\partial r}-2\mu\left\{\frac{\partial}{\partial\theta}(\omega_\phi\sin\theta)-\frac{\partial\omega_\theta}{\partial\phi}\right\}&=0,
\end{aligned}\right\}\tag{1}$$

where†

$$\Delta=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2u_r\sin\theta)+\frac{\partial}{\partial\theta}(ru_\theta\sin\theta)+\frac{\partial}{\partial\phi}(ru_\phi)\right\},\tag{2}$$

and

$$\left.\begin{aligned}
2\omega_r&=\frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial\theta}(ru_\phi\sin\theta)-\frac{\partial}{\partial\phi}(ru_\theta)\right\},\\
2\omega_\theta&=\frac{1}{r\sin\theta}\left\{\frac{\partial u_r}{\partial\phi}-\frac{\partial}{\partial r}(ru_\phi\sin\theta)\right\},\\
2\omega_\phi&=\frac{1}{r}\left\{\frac{\partial}{\partial r}(ru_\theta)-\frac{\partial u_r}{\partial\theta}\right\},
\end{aligned}\right\}\tag{3}$$

u_r, u_θ, u_ϕ being the displacements in polar coordinates at the point r, θ, ϕ.

* Love, Theory of Elasticity (4th ed.), 141.
† Love, loc. cit., 56.

Assuming u_r, u_θ, u_ϕ to be inversely proportional to r, Michell* has obtained the solutions of the equations (1), and has utilised them for the determination of the distribution of stress in a cone subjected to the action of a force at the vertex. In the present paper, I have obtained certain solutions of the equations (1), on the assumption that the displacement is inversely proportional to r^2, and with the help of these solutions, I have determined the distribution of stress in a cone to which a couple is applied at the vertex.

2. We know that the radial components of the displacement and rotation and the dilatation satisfy the equations†

$$\left.\begin{aligned} \mu\nabla^2(ru_r)+(\lambda+\mu)r\frac{\partial\Delta}{\partial r}-2\mu\Delta &= 0,\\ \nabla^2\Delta=0,\quad \nabla^2(r\omega_r) &= 0. \end{aligned}\right\} \tag{4}$$

Let us now assume that

$$u_r = \frac{F_1(\theta)}{r^2}, \quad u_\theta = \frac{F_2(\theta)}{r^2}, \quad u_\phi = \frac{F_3(\theta)}{r^2}, \tag{5}$$

where F_1, F_2, F_3 are functions of θ alone.

From (2), (3) and (5), we get

$$\Delta = \frac{F(\theta)}{r^3}, \quad 2\omega_r = \frac{f(\theta)}{r^3}, \tag{6}$$

where

$$\left.\begin{aligned} F(\theta)\sin\theta &= \frac{\partial}{\partial\theta}(F_2\sin\theta),\\ f(\theta)\sin\theta &= \frac{\partial}{\partial\theta}(F_3\sin\theta). \end{aligned}\right\} \tag{7}$$

Now, since

$$\nabla^2(r\omega_r) = 0,$$

we must have

$$\nabla^2\left[\frac{1}{r^2}f(\theta)\right] = 0,$$

so that

$$f(\theta) = A'P_1(\mu')+B'Q_1(\mu'),$$

* Proc. Lond. Math. Soc., 32 (1900), 23.

† Love, *loc. cit.*, 141.

where P_1 and Q_1 are Legendre's functions of degree 1, of the first and the second kinds respectively, and $\mu' = \cos\theta$.

But as

$$P_1(\mu') = \mu' \text{ and } Q_1(\mu') = \tfrac{1}{2}\mu' \log \frac{1+\mu'}{1-\mu'} - 1,$$

we have on putting $\mu' = \cos\theta$,

$$f(\theta) = A'\cos\theta + B'\{\cos\theta \log\cot\frac{\theta}{2} - 1\},$$

and if we want only those integrals which are regular on the axis $\theta=0$, we take $B' = 0$, so that

$$f(\theta) = A'\cos\theta. \tag{8}$$

Substituting in the second equation of (7), and integrating, we find

$$F_3 \sin\theta = -\tfrac{1}{2}A'\cos^2\theta + C'.$$

Now since u_ϕ and therefore F_3 are regular on the axis $\theta = 0$, we must take $C' = \frac{1}{2}A'$, so that

$$F_3 = \tfrac{1}{2}A'\sin\theta; \tag{9}$$

To find F_2, we observe that Δ satisfies the second equation of (4), and as

$$\Delta = \frac{1}{r^3}F(\theta),$$

we must have, if we consider only the solution regular on the axis $\theta = 0$,

$$F(\theta) = AP_2(\mu'),$$

where P_2 is Legendre's function of the first kind of degree 2.

Substituting this value of $F(\theta)$ in the first equation of (7), and intergrating we find

$$F_2 \sin\theta = -\tfrac{1}{2}A\cos^3\theta + \tfrac{1}{2}A\cos\theta + C,$$

and since F_2 is regular on the axis $\theta = 0$, we take $C = 0$, so that

$$F_2 = \tfrac{1}{2}A\sin\theta\cos\theta. \tag{10}$$

To determine F_1, we substitute the value of u_r from (5), in the first equation of (4), which becomes after substitution of the value of Δ and simplification,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_1}{\partial\theta}\right) = \frac{3\lambda+5\mu}{\mu}F(\theta), \tag{11}$$

and as $F(\theta)$ satisfies the equation,

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right)+6F=0,$$

we see that a particular integral of (11) is

$$F_1(\theta)=-\frac{3\lambda+5\mu}{6\mu}.\ F(\theta).$$

To this, we must add the solution of the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F_1}{\partial\theta}\right)=0,$$

which is $$F_1=D\log\tan\frac{\theta}{2}+E.$$

Hence, if we take $F_1(\theta)$ to be regular on the axis $\theta=0$, we have

$$F_1(\theta)=-\frac{3\lambda+5\mu}{6\mu}F(\theta)+E$$

$$=-\frac{3\lambda+5\mu}{4\mu}A\cos^2\theta+E', \qquad (12)$$

where E' is a new constant.

Therefore we have

$$\left.\begin{aligned} u_r &= \frac{1}{r^2}\left\{-\frac{3\lambda+5\mu}{4\mu}A\cos^2\theta+E'\right\},\\ u_\theta &= \frac{A}{2r^2}\sin\theta\cos\theta,\\ u_\phi &= \frac{A'}{2r^2}\sin\theta,\end{aligned}\right\}. \qquad (13)$$

Now, as the solution (13) is obtained from (4) and as some solutions of (4) correspond with stresses that require body forces for their maintenance,* therefore, we must verify that the solution (13) actually satisfies (1).

Calculating Δ, ω_r, ω_θ, ω_ϕ from (2), (3) and (13) and substituting in (1), we find that the equations (1) are identically satisfied.

* Michell, Proc. Lond. Math. Soc., 32 (1900), 24.

8. Let us assume

$$u_r = \frac{\cos\phi}{r^2} F_1(\theta), \quad u_\theta = \frac{\cos\phi}{r^2} F_2(\theta), \quad u_\phi = \frac{\sin\phi}{r^2} F_3(\theta), \tag{14}$$

where F_1, F_2, F_3 are functions of θ alone. Then we have from (2) and (3),

$$\Delta = \frac{\cos\phi}{r^3} F(\theta), \qquad 2\omega_r = \frac{\sin\phi}{r^3} f(\theta), \tag{15}$$

where

$$\left.\begin{aligned} F(\theta)\sin\theta &= \frac{\partial}{\partial\theta}(F_2 \sin\theta) + F_3, \\ f(\theta)\sin\theta &= \frac{\partial}{\partial\theta}(F_3 \sin\theta) + F_2. \end{aligned}\right\} \tag{16}$$

Now from (4) and (15), we get

$$\nabla^2\left[\frac{\cos\phi}{r^3} F(\theta)\right] = 0, \qquad \nabla^2\left[\frac{\sin\phi}{r^3} f(\theta)\right] = 0.$$

The solutions of these equations regular on the axis $\theta = 0$, are respectively,.

$$F(\theta) = A\, P_2^1(\mu'), \quad f(\theta) = B\, P_1^1(\mu'),$$

where $\mu' = \cos\theta$ and P_1^1, P_2^1 are associated Legendre's functions of the first kind of order 1 and degrees 1 and 2 respectively. They are given by

$$P_1^1(\mu') = (1-\mu'^2)^{\frac{1}{2}} \frac{dP_1(\mu_1)}{d\mu'} = (1-\mu'^2)^{\frac{1}{2}} = \sin\theta,$$

$$P_2^1(\mu') = (1-\mu'^2)^{\frac{1}{2}} \frac{dP_2(\mu')}{d\mu'} = 3\mu'(1-\mu'^2)^{\frac{1}{2}} = 3\sin\theta\cos\theta.$$

Hence

$$F(\theta) = 3A\sin\theta\cos\theta, \quad f(\theta) = B\sin\theta. \tag{17}$$

On substitution from (17), the equations (16) become

$$\left.\begin{aligned} \frac{\partial}{\partial\theta}(F_2 \sin\theta) + F_3 &= 3A\sin^2\theta\cos\theta, \\ \frac{\partial}{\partial\theta}(F_3 \sin\theta) + F_2 &= B\sin^2\theta. \end{aligned}\right\} \tag{18}$$

The elimination of F_2 between these two equations gives

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(F_3\sin\theta)\right]-\frac{F_3\sin\theta}{\sin^2\theta}=3(B-A)\sin\theta\cos\theta$$

$$=(B-A)\,P_2^1(\cos\theta). \tag{19}$$

Comparing this with the equation satisfied by $P_2^1(\cos\theta)$, *viz.*, the equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\left(6-\frac{1}{\sin^2\theta}\right)V=0, \tag{20}$$

it is obvious that a particular integral of (19) is

$$F_3\sin\theta=-\tfrac{1}{6}(B-A)\,P_2^1(\cos\theta)=-\tfrac{1}{2}(B-A)\sin\theta\cos\theta.$$

To this we must add the solution of

$$\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(F_3\sin\theta)\right]-F_3\sin\theta=0,$$

or
$$\frac{d^2}{dt^2}(F_3\sin\theta)-F_3\sin\theta=0,$$

where
$$t=\int\frac{d\theta}{\sin\theta}=\log\tan\frac{\theta}{2}.$$

The solution is

$$F_3\sin\theta=G\,e^{t}+H\,e^{-t}$$

$$=G\tan\frac{\theta}{2}+H\cot\frac{\theta}{2}.$$

Omitting the term in H, since we are seeking only those solutions regular the axis $\theta=0$, we get as a solution of (19),

$$F_3\sin\theta=-\tfrac{1}{2}(B-A)\sin\theta\cos\theta+G\tan\frac{\theta}{2},$$

so that
$$F_3=-\tfrac{1}{2}(B-A)\cos\theta+\tfrac{1}{2}G\sec^2\frac{\theta}{2}. \tag{21}$$

The second equation of (18), then gives

$$F_2=\tfrac{1}{2}(B-A)+A\sin^2\theta-\tfrac{1}{2}G\sec^2\frac{\theta}{2}. \tag{22}$$

To obtain F_1, we substitute from (14) and (15) in the first equation of (4) and we have

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial F_1}{\partial\theta}\right)-\frac{F_1}{\sin^2\theta}=\frac{3\lambda+5\mu}{\mu}\,F(\theta). \tag{23}$$

Remembering that F satisfies the equation (20), we have as a particular integral of (23),

$$F_1=-\frac{3\lambda+5\mu}{6\mu}F(\theta)$$

$$=-\frac{3\lambda+5\mu}{2\mu}A\sin\theta\cos\theta,$$

on substitution from (17).

To this we must add the solution of

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial F_1}{\partial\theta}\right)-\frac{F_1}{\sin^2\theta}=0$$

which is regular on the axis $\theta=0$, and this solution has already been found to be $\tan\frac{\theta}{2}$.

Hence

$$F_1=-\frac{3\lambda+5\mu}{2\mu}A\sin\theta\cos\theta+P\tan\frac{\theta}{2}. \tag{24}$$

Therefore we have

$$\left.\begin{aligned}
u_r&=\left[-\frac{3\lambda+5\mu}{2\mu}A\sin\theta\cos\theta+P\tan\frac{\theta}{2}\right]\frac{\cos\phi}{r^2},\\
u_\theta&=\left[C+A\sin^2\theta-\tfrac{1}{2}G\sec^2\frac{\theta}{2}\right]\frac{\cos\phi}{r^2},\\
u_\phi&=\left[-C\cos\theta+\tfrac{1}{2}G\sec^2\frac{\theta}{2}\right]\frac{\sin\phi}{r^2},
\end{aligned}\right\} \tag{25}$$

where C is put for $\frac{1}{2}(B-A)$.

Calculating Δ, ω_r, ω_θ, ω_ϕ from (25) and substituting in the equation (1), we find that they are identically satisfied, if $G=P$.

4. In polar co-ordinates, the components of strain are given by*

$$\left.\begin{aligned}
e_{rr} &= \frac{\partial u_r}{\partial r}, \quad e_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}, \\
e_{\phi\phi} &= \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta}{r}\cot\theta + \frac{u_r}{r}, \\
e_{\theta\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial \theta} - u_\phi \cot\theta\right) + \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial \phi}, \\
e_{\phi r} &= \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}, \\
e_{r\theta} &= \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}.
\end{aligned}\right\} \qquad (26)$$

Further, we have the stress-strain relations

$$\left.\begin{aligned}
\widehat{rr} &= \lambda\Delta + 2\mu e_{rr}, & \widehat{\theta\theta} &= \lambda\Delta + 2\mu e_{\theta\theta}, \\
\widehat{\phi\phi} &= \lambda\Delta + 2\mu e_{\phi\phi}, & \widehat{r\theta} &= \mu e_{r\theta}, \\
\widehat{\theta\phi} &= \mu e_{\theta\phi}, & \widehat{\phi r} &= \mu e_{\phi r}.
\end{aligned}\right\} \qquad (27)$$

The direction-cosines of the directions of r, θ, ϕ are respectively

$(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$, $(\cos\theta\cos\phi,\ \cos\theta\sin\phi,\ -\sin\theta)$, $(-\sin\phi,\ \cos\phi,\ 0)$.

Now, if we take the body to be a cone bounded by $\theta = \alpha$, and if X, Y, Z, L, M, N, be the components of the resultant traction across the portion of a small sphere r, bounded by the cone, the centre of the sphere being at the vertex of the cone, we have

$$\left.\begin{aligned}
X &= \iint (\widehat{rr}.\sin\theta\cos\phi + \widehat{r\theta}.\cos\theta\cos\phi - \widehat{r\phi}.\sin\phi)\, r^2 \sin\theta\, d\theta\, d\phi, \\
Y &= \iint (\widehat{rr}.\sin\theta\sin\phi + \widehat{r\theta}.\cos\theta\sin\phi + \widehat{r\phi}\cos\phi) r^2 \sin\theta\, d\theta\, d\phi, \\
Z &= \iint (\widehat{rr}.\cos\theta - \widehat{r\theta}\sin\theta) r^2 \sin\theta\, d\theta\, d\phi,
\end{aligned}\right\} \qquad (28a)$$

* Love, *loc. cit.*, 56.

$$\left.\begin{aligned} L &= \iint(-\widehat{r\theta}.\, r\sin\phi - \widehat{r\phi}\cos\phi.r\cos\theta)r^2\sin\theta d\theta d\phi,\\ M &= \iint(\widehat{r\theta}.\, r\cos\phi - \widehat{r\phi}\sin\phi.r\cos\theta)r^2\sin\theta d\theta d\phi,\\ N &= \iint\widehat{r\phi}.\, r\sin\theta.\, r^2\sin\theta d\theta d\phi, \end{aligned}\right\} \qquad (28b)$$

the limits of θ being 0 and a and those of ϕ, 0 and 2π.

5. Let us now consider the solution (18). Substituting in (26), we get

$$e_{rr} = \frac{1}{r^3}\left\{\frac{3\lambda+5\mu}{2\mu}A\cos^2\theta - 2E'\right\},$$

$$e_{\theta\theta} = \frac{1}{r^3}\left\{-\frac{3\lambda+\mu}{4\mu}A\cos^2\theta + E' - \tfrac{1}{2}A\right\},$$

$$e_{\phi\phi} = \frac{1}{r^3}\left\{-\frac{3(\lambda+\mu)}{4\mu}A\cos^2\theta + E'\right\},$$

$$e_{\theta\phi} = 0,\quad e_{\phi r} = -\frac{3}{2}.\frac{A'}{r^3}\sin\theta,$$

$$e_{r\theta} = \frac{3\lambda+2\mu}{2\mu}.\frac{A}{r^3}\sin\theta\cos\theta.$$

Hence

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{r^3}\left\{\frac{9\lambda+10\mu}{2}A\cos^2\theta - \lambda A - 4\mu E'\right\},\\ \widehat{\theta\theta} &= \frac{1}{r^3}\left\{-\tfrac{1}{2}\mu A\cos^2\theta - \tfrac{1}{2}(\lambda+2\mu)A + 2\mu E'\right\},\\ \widehat{\phi\phi} &= \frac{1}{r^3}\left\{-\frac{3}{2}\mu A\cos^2\theta - \tfrac{1}{2}\lambda A + 2\mu E'\right\},\\ \widehat{\theta\phi} &= 0,\ \widehat{\phi r} = -\frac{3}{2}\frac{\mu A'}{r^3}\sin\theta,\\ \widehat{r\theta} &= \frac{3\lambda+2\mu}{2}.\frac{A}{r^3}\sin\theta\cos\theta. \end{aligned}\right\} \qquad (29)$$

If the boundary of the cone be free from tractions,

$$\widehat{r\theta} = \widehat{\theta\theta} = \widehat{\theta\phi} = 0, \quad \text{when } \theta = a,$$

which gives $A = 0$, $E' = 0$.

Putting $A = 0$, $E' = 0$, in (29) and then substituting in (28*a*) and (28*b*), we get

$$\left.\begin{aligned} &X = Y = Z = L = M = 0, \\ &N = -\pi\mu A'(1-\cos a)(1-\cos a+\sin^2 a), \end{aligned}\right\} \tag{30}$$

so that A' is determined when N is given.

Therefore

$$u_r = 0, \quad u_\theta = 0, \quad u_\phi = \frac{A'\sin\theta}{2r^2}, \tag{31}$$

where A' is given by the last equation of (30), solves the problem of a cone subjected to the action of a couple N at the vertex, about the axis of the cone.

In this case

$$\widehat{rr} = 0, \quad \widehat{\theta\theta} = 0, \quad \widehat{\phi\phi} = 0,$$

$$\widehat{r\theta} = 0, \quad \widehat{\theta\phi} = 0, \quad \widehat{\phi r} = -\frac{3\lambda A'}{2r^3}\sin\theta.$$

6. Let us next consider the solution (25). Substituting in (26), we get

$$e_{rr} = \left\{\frac{3\lambda+5\mu}{\mu} A\sin\theta\cos\theta - 2P\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$e_{\theta\theta} = \left\{-\frac{3\lambda+\mu}{2\mu} A\sin\theta\cos\theta + \left(P - \tfrac{1}{2}G\sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$e_{\phi\phi} = \left\{-\frac{3(\lambda+\mu)}{2\mu} A\sin\theta\cos\theta + \left(P + \tfrac{1}{2}G\sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$e \;\; = \left\{-A\sin\theta + G\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}\right\}\frac{\sin\phi}{r^3},$$

$$e_{\phi r} = \left\{\frac{3\lambda+5\mu}{2\mu}\,A\cos\theta + 3C\cos\theta - (\tfrac{1}{2}P + \tfrac{3}{2}G)\sec^2\frac{\theta}{2}\right\}\frac{\sin\phi}{r^3},$$

$$e_{r\theta} = \left\{-3C - \tfrac{3}{2}A - \frac{3\lambda+2\mu}{2\mu}\,A\cos 2\theta + (\tfrac{1}{2}P + \tfrac{3}{2}G)\sec^2\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3}.$$

Hence

$$\widehat{rr} = \left\{(9\lambda+10\mu)A\sin\theta\cos\theta - 4\mu P\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$\widehat{\theta\theta} = \mu\left\{-A\sin\theta\cos\theta + \left(2P - G\sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$\widehat{\phi\phi} = \mu\left\{-3A\sin\theta\cos\theta + \left(2P + G\sec^2\frac{\theta}{2}\right)\tan\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3},$$

$$\widehat{\theta\phi} = \mu\left\{-A\sin\theta + G\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}\right\}\frac{\sin\phi}{r^3},$$

$$\widehat{\phi r} = \mu\left\{\frac{3\lambda+5\mu}{2\mu}A\cos\theta + 3C\cos\theta - (\tfrac{1}{2}P + \tfrac{3}{2}G)\sec^2\frac{\theta}{2}\right\}\frac{\sin\phi}{r^3},$$

$$\widehat{r\theta} = \mu\left\{-3C - \tfrac{3}{2}A - \frac{3\lambda+2\mu}{2\mu}A\cos 2\theta + (\tfrac{1}{2}P + \tfrac{3}{2}G)\sec^2\frac{\theta}{2}\right\}\frac{\cos\phi}{r^3}.$$

If the surface of the cone be free from tractions, we have, when $\theta = \alpha$,

$$\widehat{r\theta} = \widehat{\theta\theta} = \widehat{\theta\phi} = 0.$$

Hence, remembering that $G = P$, we get

$$-3C - \tfrac{3}{2}A - \frac{3\lambda+2\mu}{2\mu}A\cos 2\alpha + 2P\sec^2\frac{\alpha}{2} = 0,$$

$$-A\sin\alpha\cos\alpha + P\left(2 - \sec^2\frac{\alpha}{2}\right)\tan\frac{\alpha}{2} = 0,$$

$$-A\sin\alpha + P\sec^2\frac{\alpha}{2}\tan\frac{\alpha}{2} = 0,$$

from which we get

$$\left.\begin{aligned} G &= P = 2A\cos^4\frac{\alpha}{2}, \\ C &= \left(-\tfrac{1}{2} - \frac{3\mu+2\mu}{6\mu}\cos 2\alpha + \tfrac{1}{3}\cos^2\frac{\alpha}{2}\right)A, \end{aligned}\right\} \tag{32}$$

so that all the constants are expressed in terms of A.

To calculate the resultant traction across a spherical section r of the cone, we substitute in (28a) and (28b) and find

$$\left.\begin{aligned} &X = Y = Z = L = N = 0, \\ &M = -\pi A(1-\cos\alpha)^2\left[2\{\lambda(1+\cos\alpha)+\mu(1+\tfrac{1}{3}\cos\alpha)\}(1+\cos\alpha)\right. \\ &\qquad \left. + (\lambda+\tfrac{2}{3}\mu)\cos^3\alpha\right], \end{aligned}\right\} \tag{33}$$

which determines A when M is known.

Therefore (25) with (32) and (33) solves the problem of a cone subjected to a couple at the vertex about an axis perpendicular to the axis of the cone.

7. If we put $\alpha = \frac{\pi}{2}$, we pass from a cone to an infinite solid bounded by a plane.

When the axis of the couple is normal to the plane boundary,

$$u_r = 0, \quad u_\theta = 0, \quad u_\phi = -\frac{N}{4\pi\mu}\cdot\frac{\sin\theta}{r^2},$$

$$\widehat{rr} = \widehat{\theta\theta} = \widehat{\phi\phi} = \widehat{r\theta} = \widehat{\theta\phi} = 0, \quad \widehat{\phi r} = \frac{3N}{4\pi}\cdot\frac{\sin\theta}{r^3},$$

where N is the magnitude of the couple applied.

This solution can very easily be identified* with the solution due to a centre of rotation at the origin about the axis of z.

* Love, *loc. cit.*, 187.

When the axis of the couple lies in the plane boundary,

$$u_r = \left[-\frac{3\lambda+5\mu}{2\mu}\sin\theta\cos\theta + \tfrac{1}{2}\tan\frac{\theta}{2} \right] \frac{A\cos\phi}{r^2},$$

$$u_\theta = \left[\frac{\lambda+\mu}{2\mu} + \sin^2\theta - \tfrac{1}{2}\sec^2\frac{\theta}{2} \right] \frac{A\cos\phi}{r^2},$$

$$u_\phi = \left[-\frac{\lambda+\mu}{2\mu}\cos\theta + \tfrac{1}{2}\sec^2\frac{\theta}{2} \right] \frac{A\sin\phi}{r^2},$$

where

$$A = -\frac{M}{2\pi(\lambda+\mu)},$$

M being the magnitude of the couple applied.

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À PROPOS D'UN MÉMOIRE DU REGRETTÉ GANESH PRASAD

Ce mémoire porte le titre "Failure of Lebesgue's criterion" qui fait allusion au critère de sommabilité, généralisation de celui de Fejér, que j'ai donné.

Le titre du mémoire incite à penser que j'ai commis une erreur ; c'est ce qu'a cru tout naturellement M. le Dr. S. C. Bagchi quand il écrivait son bel article : In memoriam, Dr. Ganesh Prasad, (ce Bulletin, Vol. XXVII, Nos. 1, 2, 1935). En fait, je n'ai, au sujet du critère en question, commis aucune erreur et n'ai donc pas eu à reconnaître mon erreur.

Que faisait le Dr. Prasad dans son article ? Il étudiait, avec son talent ordinaire, la sommabilité de certaines séries de Fourier, sommabilité que n'aurait pu revéler mon critère. Mais ceci n'est pas une "failure" du critère. Moi-même, dès le début, j'avais signalé des cas de sommabilité que n'aurait pu révéler mon critère. Ce critère a toujours été donné par moi comme suffisant, et non comme nécessaire.

Parceque une série peut être convergente sans que le rapport $\frac{u_{n+1}}{u_n}$ ait une limite inférieure à un, parle-t-on de la "failure" du critère de D'Alembert qui affirme la convergence quand $\frac{u_{n+1}}{u_n}$ a une limite inférieure à un ?

Je profite de l'occasion de cette rectification pour m'associer aux regrets qu'a causés la mort du Dr. Ganesh Prasad et pour dire toute la haute estime en laquelle j'ai toujours tenu votre savant surpatriote, et à cause de ses travaux et à cause de son action efficace pour la création d'un centre actif de recherches mathématiques aux Indes.

HENRI LEBESGUE.

A Correction

In Memoriam

(Dr. Ganesh Prasad)

In the obituary on Ganesh Prasad, there is an unfortunate oversight pointed out to me by Professor Lebesgue. I have stated that "there was an oversight which was detected by Prasad and when communicated to Lebesgue himself, the latter owned his mistake." As a matter of fact, Prof. Lebesgue has committed no mistake, as he points out in a note, published in this issue. I regret this statement and withdraw it, with apologies to Prof. Lebesgue.

S. C. Bagchi.

10

On Schläfli's generalization of Napier's Pentagramma Mirificum

By

H. S. M. Coxeter.

In 1614, Napier showed that any right-angled spherical triangle can be regarded as belonging to a cycle of five triangles with closely related elements.* In fact, if the given triangle is ABC, right-angled at C, and we write

$$\nu_0 = A, \quad \nu_1 = \tfrac{1}{2}\pi - a, \quad \nu_2 = c, \quad \nu_3 = \tfrac{1}{2}\pi - b, \quad \nu_4 = B;$$

then the remaining triangles are obtained by cyclic permutation of the suffix numbers 0, 1, 2, 3, 4.

If any five collinear points are denoted by X_0, X_1, X_2, X_3, X_4 (in this order), the semi-circles on diameters X_1X_3, X_2X_4, X_0X_3, X_1X_4, X_0X_2 intersect to form a curvilinear pentagon (Fig. iii) whose angles are the values of $2\nu_0$, $2\nu_1$, $2\nu_2$, $2\nu_3$, $2\nu_4$ for some right-angled spherical triangle. (This remarkable theorem is due to Dr. G. T. Bennett, who will publish it in greater detail elsewhere.) If we make X_0 and X_4 coincide at infinity, we are left with the semi-circle on X_1X_3 and the perpendicular line at X_2. This diagram can be regarded as representing a *plane* right-angled triangle of sides $(X_1X_2)^{\frac{1}{2}}$, $(X_2X_3)^{\frac{1}{2}}$, $(X_1X_3)^{\frac{1}{2}}$. (See Fig. i.) Of course, in this

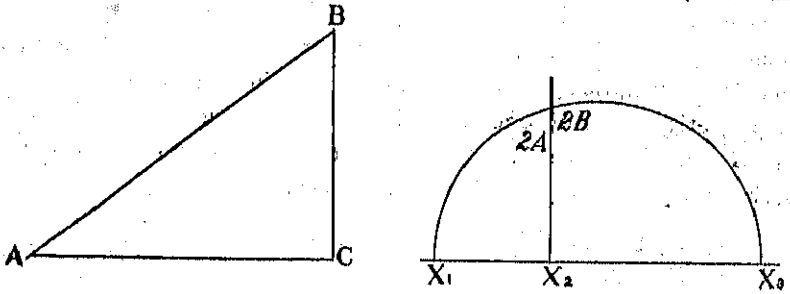

Fig. i.

* Napier, 1.

simplest case the diagram is more complicated than the figure itself ; the important point is that, when the right-angled triangle is replaced by an analogous figure in Euclidean or non-Euclidean space of any number of dimensions, the diagram remains in the Euclidean plane.

In 1906, Wythoff extended Napier's result to spherical space of three dimensions, considering a "double-rectangular" tetrahedron.* Such a tetrahedron is bounded by four right-angled triangles. It has six dihedral angles, of which three are right ; twelve face-angles, of which four are right and two are equal to dihedral angles ; and six edges—$3+6+6=15$ "elements," in all. If we name these fifteen elements $\nu_{01}, \nu_{02}, \ldots, \nu_{45}$ in a special order,** and make the convention

$$\nu_{rs}' = \tfrac{1}{2}\pi - \nu_{sr},$$

then any formula connecting them remains true when we increase (or diminish) all suffixes by 1 (mod 6) and at the same time transpose the two suffixes of each ν.

It is natural to put these ideas together in the following manner. The semi-circles whose diameters are determined by six collinear points $X_0, X_1, \ldots, X_5$, intersect in just fifteen ways ; we find that the fifteen angles so determined are the values of $2\nu_{01}, 2\nu_{02}, \ldots, 2\nu_{45}$ for some spherical tetrahedron. *E.g.*, $2\nu_{01}$ is the angle between the semi-circles whose diameters are $X_2 X_4, X_3X_5$.

The analogous simplex in spherical space of $m-1$ dimensions has $\binom{m+2}{4}$ elements, which may be denoted by

$$[r\,s\,t\,u] \qquad (0 \leqq r < s < t < u \leqq m+1).$$

If $m+2$ collinear points are called $X_0, X_1, \ldots, X_{m+1}$, then $2[rstu]$ is the angle between the semi-circles whose diameters are X_rX_t, X_sX_u. Another way of stating the same result is that the squared trigonometric functions of $[rstu]$ are the cross-ratios of the four points X_r, X_s, X_t, X_u. Thus a simplex (of this special kind), in spherical or elliptic space of any number of dimensions, is represented on projective space of one dimension. By varying the order in which the X's are numbered, we obtain an analogous representation for such a simplex in any non-Euclidean space.

* Wythoff, 1.

** Namely,

$$\alpha_{12} = \nu_{45}, \qquad \alpha_{23} = \nu_{05}, \qquad \alpha_{34} = \nu_{01},$$
$$A_{1s} = \nu_{0s} \quad (s=2, 3, 4), \qquad A_{4r} = \nu_{r5} \quad (r=1, 2, 3),$$
$$a_{rs} = \nu_{tu} \qquad (r\,s\,t\,u \text{ any permutation of } 1\,2\,3\,4 \,;\, t < u).$$

1. *Dr. Bennett's representation of the Pentagramma Mirificum by five collinear or concyclic points.*

Let ABC be a spherical triangle, right-angled at C. The *Pentagramma Mirificum** is a cycle of five great circles: those containing the sides a, c, b of the triangle, and the polar circles of the vertices B, A. In one order of arrangement, every consecutive pair of circles are orthogonal; in another, every non-consecutive pair are orthogonal. We prefer the latter order; the consecutive circles then form a pentagon in which each side lies in the polar circle of the opposite vertex.

The angles between pairs of these five great circles are the same as the angles between the planes in which they lie, and these will not be altered if we shift two of the planes (by translation) so as to destroy the concurrence. We are thus led to consider a cycle of five planes *0, 1, 2, 3, 4* (in ordinary space), such that the pairs *02, 24, 41, 13, 30* are each perpendicular. Any four of these planes, say *0, 1, 2, 3*, form a tetrahedron of the kind that Wythoff calls *double-rectangular*. (See Fig. ii.) If A_0, A_1, A_2, A_3 are the vertices opposite to the faces *0, 1, 2, 3*, it is clear that the edges A_0A_1, A_1A_2, A_2A_3, are all perpendicular. In other words, the four faces $A_1A_2A_3$, $A_0A_2A_3$, $A_0A_1A_3$, $A_0A_1A_2$ are plane right-angled triangles.

Let X_0, X_1, X_2, X_3 be four points in any straight line, so arranged that $X_0X_1 = (A_0A_1)^2$, $X_1X_2 = (A_1A_2)^2$, $X_2X_3 = (A_2A_3)^2$.

Then (by Pythagoras) we shall also have

$$X_0X_2 = (A_0A_2)^2, \quad X_0X_3 = (A_0A_3)^2, \quad X_1X_3 = (A_1A_3)^2.$$

Following Schoute and Wythoff, we use the following notation for the angles of a tetrahedron: α_{rs} is the dihedral angle at the edge opposite to A_rA_s (*i. e.*, the angle between the planes r, s), and A_{rs} is the angle at A_r in the face opposite to A_s.

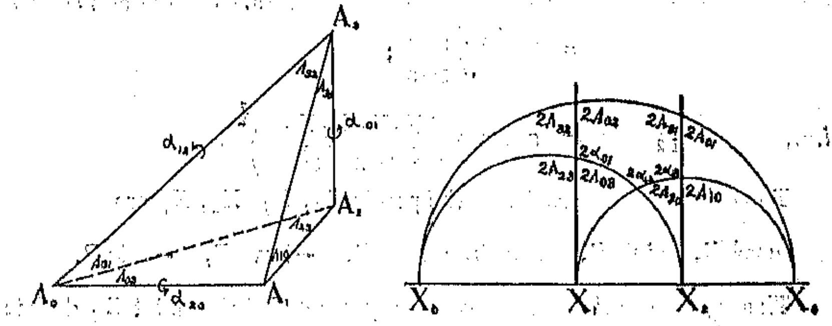

Fig. ii.

* Gauss, 1. See also Sommerville, 1, 118 (Fig. 74).

Clearly,

$$\alpha_{02} = \alpha_{03} = \alpha_{13} = A_{12} = A_{13} = A_{20} = A_{21} = \tfrac{1}{2}\pi,$$

$$\alpha_{01} = A_{23} = \text{arc}\cos(A_1A_2/A_0A_2) = \text{arc}\cos(X_1X_2/X_0X_2)^{\frac{1}{2}},$$

$$\alpha_{23} = A_{10} = \text{arc}\cos(A_1A_2/A_1A_3) = \text{arc}\cos(X_1X_2/X_1X_3)^{\frac{1}{2}}.$$

If $rstu$ is a permutation of $0\ 1\ 2\ 3$, such that $r < s < t$, then

$$A_{ru} = \text{arc}\cos(A_rA_s/A_rA_t) = \text{arc}\cos\,(X_rX_s/X_rX_t)^{\frac{1}{2}},$$

$$A_{tu} = \tfrac{1}{2}\pi - A_{ru}.$$

The only angle that remains to be calculated is α_{12}.

On a sphere with centre A_0, the faces *1*, *2*, *3* cut out a right-angled triangle ABC, such that

$$A = \alpha_{23},\quad B = \alpha_{12},\quad a = A_{01},\quad c = A_{02},\quad b = A_{03}.$$

Conversely, any right-angled spherical triangle ABC can be interpreted as a trihedral angle (A_0, ABC), and we obtain the fourth face of the tetrahedron $A_0A_1A_2A_3$ by taking an arbitrary plane perpendicular to A_0A_1. The above formulae show that

$$\begin{aligned}\cos A = \cos\alpha_{23} &= A_1A_2/A_1A_3\\ &= (A_1A_2/A_0A_1)\,(A_0A_1/A_1A_3) = \tan A_{03}\cot A_{02}\\ &= \tan b\cot c.\end{aligned}$$

(This is, of course, one of Napier's rules; but there is a certain satisfaction in deducing it *ab initio*.) Interchanging A and B, a and b, we have

$$\begin{aligned}\cos B = \tan a\cot c = \tan A_{01}\cot A_{02} &= (A_2A_3/A_0A_2)\,(A_0A_1/A_1A_3)\\ &= \frac{A_0A_1.A_2A_3}{A_0A_2.A_1A_3};\end{aligned}$$

i.e., $$\alpha_{12} = B = \text{arc}\cos\left(\frac{X_0X_1.X_2X_3}{X_0X_2.X_1X_3}\right)^{\frac{1}{2}}.$$

Thus the six cross-ratios of the points X_0, X_1, X_2, X_3 are

$$\cos^2 B,\quad \sin^2 B,\quad \sec^2 B,\ \operatorname{cosec}^2 B,\quad -\tan^2 B,\quad -\cot^2 B.$$

This representation becomes more symmetrical if we let X_4 denote the point at infinity in the line of the X's, and then perform a projection or inversion, so as to make X_4 accessible without altering

cross-ratios. We now have

$$
(1\cdot1)\qquad \begin{cases} A = a_{23} = [1\ 2\ 3\ 4], \\ a = A_{01} = [0\ 2\ 3\ 4], \\ c = A_{02} = [0\ 1\ 3\ 4], \\ b = A_{03} = [0\ 1\ 2\ 4], \\ B = a_{12} = [0\ 1\ 2\ 3], \end{cases}
$$

where

$$
(1\cdot2)\qquad [r\ s\ t\ u] = \operatorname{arc\,cos}\left(\frac{X_rX_s.X_tX_u}{X_rX_t.X_sX_u}\right)^{\frac{1}{2}},
$$

X_0, X_1, X_2, X_3, X_4 being five collinear points. Still more symmetrically, restoring the fifth plane 4 (perpendicular to A_0A_3), we have

$$
(1\cdot3)\qquad \begin{cases} A = a_{23} = [1\ 2\ 3\ 4], \\ \tfrac{1}{2}\pi - a = a_{34} = [2\ 3\ 4\ 0], \\ c = a_{40} = [3\ 4\ 0\ 1], \\ \tfrac{1}{2}\pi - b = a_{01} = [4\ 0\ 1\ 2], \\ B = a_{12} = [0\ 1\ 2\ 3]. \end{cases}
$$

(This is Napier's cycle of angles.)

We next deduce a representation in the Euclidean plane. We draw semi-circles on X_0X_2, X_0X_3, X_1X_3, X_1X_4, X_2X_4 as diameters, and observe that the semi-circles on X_rX_t and X_sX_u $(r < s < t < u)$ meet at angle $2\,[rstu]$. (This becomes obvious when we transform one of the circles into a straight line by inversion about one of the points. See Fig. i.) Thus the five "elements" of a right-angled spherical triangle are halves of the angles between pairs of five such semi-circles. (See Fig. iii. Fig. ii shows the same with X_4 at infinity.)

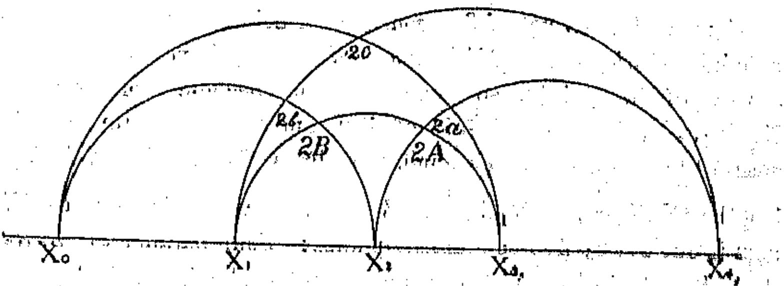

Fig. iii.

Finally, after inversion with respect to a general circle in the plane, we have a perfectly symmetrical representation for the angles between the five planes $0, 1, 2, 3, 4,$ as halves of the angles between five circles orthogonal to one circle, as in Fig. iv. (1·2) continues to hold, if X_rX_s means the *chord* joining X_r to X_s. The *sign* of such a chord agrees with one of the arcs joining the same two points, namely, that one which avoids a certain "barrier point," arbitrarily assigned once for all. (With this convention, Ptolemy's Theorem takes the symmetrical form

$$X_0X_1.X_2X_3+X_0X_2.X_3X_1+X_0X_3.X_1X_2 = 0.)$$

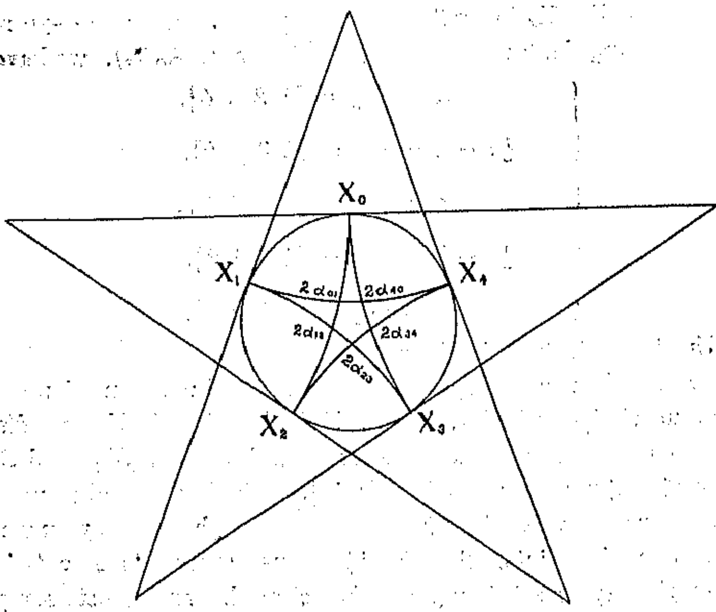

Fig. iv.

Conversely, any set of five concyclic or collinear points represents a definite *Pentagramma Mirificum*. We merely have to invert about any one of the points, so that the rest become collinear points X_0, X_1, X_2, X_3 (with X_4 at infinity), and then draw (in Euclidean space) three mutually perpendicular lines A_0A_1, A_1A_2, A_2A_3, of lengths $(X_0X_1)^{\frac{1}{2}}$, $(X_1X_2)^{\frac{1}{2}}$, $(X_2X_3)^{\frac{1}{2}}$. The *Pentagramma Mirificum* is cut out by five concurrent planes: Four parallel to $A_1A_2A_3$, $A_0A_2A_3$, $A_0A_1A_3$, $A_0A_1A_2$, and one perpendicular to A_0A_3.

2. *The general orthoscheme.*

Consider now the general simplex $A_0A_1 \ldots\ldots A_m$ in Euclidean or non-Euclidean m-space. Following Schoute,* we write $A_s(A_p)A_t$ for the angle A_p of the triangle $A_sA_pA_t$, $A_s(A_pA_q)A_t$ for the angle a_{st} of the tetrahedron $A_sA_pA_qA_t$, and, analogously, $A_s(A_p \ldots A_q)A_t$ for the dihedral angle of the simplex $A_sA_p \ldots A_qA_t$ opposite to its edge A_sA_t, *i.e.*, for the angle between the spaces $A_sA_p \ldots A_q$, $A_p \ldots A_qA_t$. (When the symbol involves *all* the A's, this is a dihedral angle of the whole simplex.) There will be no loss of generality in supposing that $s < t$.

The m-dimensional analogue of the right-angled triangle and of the double-rectangular tetrahedron is what Schläfli calls an *orthoscheme*,** *viz.*, a simplex in which the m edges A_0A_1, A_1A_2, ... $A_{m-1}A_m$ are all perpendicular. This means that the n-space $A_0A_1 \ldots A_n$ is absolutely perpendicular to the $(m-n)$-space $A_nA_{n+1} \ldots A_m$ (for all values of n from 1 to $m-1$). It follows that the "face" of any number of dimensions, namely,

$$A_p \ldots A_q \quad (0 \leqq p < \ldots < q \leqq m),$$

is itself an orthoscheme.

The dihedral angle $A_s(A_0A_1 \ldots A_{s-1}A_{s+1} \ldots A_{t-1}A_{t+1} \ldots A_m)A_t$ is a right angle whenever $s < t-1$. For, by definition, it is the angle between the primes ...

$$A_0A_1 \ldots A_{t-1}A_{t+1} \ldots A_m, \quad A_0A_1 \ldots A_{s-1}A_{s+1} \ldots A_m.$$

If $s < n < t$, these are perpendicular, since they contain respectively the absolutely perpendicular spaces

$$A_0A_1 \ldots A_n, \quad A_nA_{n+1} \ldots A_m.$$

By restricting consideration to the orthoscheme $A_sA_p \ldots A_qA_t$, it follows that

(2·1) $$A_s(A_p \ldots A_q)A_t = \tfrac{1}{2}\pi$$

whenever any of the numbers $p, \ldots, q$ lie between s and t.

We proceed to prove that every angle $A_s(A_p \ldots A_q)A_t$ is equal to one in which the parentheses contain at most two A's. Let us

* Schoute, 1, 268.
** Schläfli, 1, 360 ; 2, 231.

suppose that q and r are either the two least or the two greatest of the numbers $s, p, \ldots, q, r, t$, and that in the former case $r < q$ ($< s$) and in the latter $r > q$ ($> t$). Then, since $A_r A_q$ is perpendicular to $A_s A_p \ldots A_q A_t$,

$$A_s(A_p \ldots A_q A_r)A_t = A_s(A_p \ldots A_q)A_t.$$

Repeated application of this formula gives the desired result. In fact, *if p is the greatest of those suffixes which are less than s, and q is the least of those which are greater than t, we have*

$$(2{\cdot}2) \qquad A_s(A_p \ldots A_q)A_t = A_s(A_p A_q)A_t \quad (p < s < t < q).$$

Moreover, if all of $p, \ldots, q$ are less than s (or greater than t), and p is the greatest (least) of them, we have

$$(2{\cdot}3) \qquad A_s(A_p \ldots A_q)A_t = A_s(A_p)A_t \quad (p < s < t \text{ or } s < t < p).$$

3. *The representation by $m + 2$ collinear or concyclic points.*

We consider in more detail the case when the m-space is Euclidean, so that any three vertices of the orthoscheme belong to a plane right-angled triangle (with its right-angle at the vertex whose suffix is intermediate). Let $X_0, X_1, \ldots, X_m$ be $m + 1$ points in any straight line, so arranged that

$$X_0X_1 = (A_0A_1)^2, \quad X_1X_2 = (A_1A_2)^2, \ldots, X_{m-1}X_m = (A_{m-1}A_m)^2.$$

Then (by Pythagoras) we shall have

$$(3{\cdot}1) \qquad X_sX_t = (A_sA_t)^2$$

for all values of s, t $(0 \leq s < t \leq m)$. Our previous work on the double-rectangular tetrahedron now shows that

$$(3{\cdot}2) \qquad A_s(A_pA_q)A_t = \text{arc}\cos\left(\frac{X_pX_s \,.\, X_tX_q}{X_pX_t \,.\, X_sX_q}\right)^{\frac{1}{2}} = [pstq]$$

$$(p < s < t < q),$$

and that

$$A_s(A_p)A_t = \text{arc}\cos(X_pX_s/X_pX_t)^{\frac{1}{2}} = [p\ s\ t\ m+1]$$

$$(p < s < t),$$

where X_{m+1} is the point at infinity in the line of the X's. These

formulae remain valid when we transform the X's by an inversion so as to render X_{m+1} accessible.

By drawing a sphere around A_0, we derive an $(m-1)$-dimensional spherical orthoscheme $B_1 \ldots B_m$, whose vertex B_p lies on A_0A_p. (It *is* an orthoscheme, since it has the same arrangement of right angles as $A_1 \ldots A_m$). Conversely, any $(m-1)$-dimensional spherical orthoscheme can be regarded as being cut out by m primes through a point A_0 in Euclidean m-space, and we can complete the Euclidean orthoscheme by taking an arbitrary plane perpendicular to A_0B_1. The edges and angles of the spherical orthoscheme $B_1 \ldots B_m$ are thus given by the formulae

$$(3{\cdot}3)\quad \begin{cases} B_sB_t = [0\ s\ t\ m+1] & (s < t), \\ B_s(B_p)B_t = [p\ s\ t\ m+1] & (p < s < t), \\ B_s(B_q)B_t = [0\ s\ t\ q] & (s < t < q), \\ B_s(B_pB_q)B_t = [p\ s\ t\ q] & (p < s < t < q), \end{cases}$$

and are halves of the angles between pairs of semi-circles determined by the $m+2$ collinear points $X_0, X_1, \ldots, X_{m+1}$.

In particular, the dihedral angles (other than those which are necessarily right angles) are:

$$B_1\,(B_3 \ldots B_m)\,B_2 = B_1\,(B_3)\,B_2 \qquad = [0\ 1\ 2\ 3],$$

$$B_2\,(B_1B_4 \ldots B_m)\,B_3 = B_2\,(B_1B_4)\,B_3 = [1\ 2\ 3\ 4],$$

$$\ldots \qquad \ldots \qquad \ldots$$

$$B_{m-2}\,(B_1 \ldots B_{m-3}B_m)\,B_{m-1} = B_{m-2}\,(B_{m-3}B_m)\,B_{m-1}$$
$$= [m-3\quad m-2\quad m-1\quad m],$$

$$B_{m-1}\,(B_1 \ldots B_{m-2})\,B_m = B_{m-1}\,(B_{m-2})\,B_m$$
$$= [m-2\quad m-1\quad m\quad m+1].$$

These lead, by cyclic permutation of the numbers $0, 1, \ldots, m+1$, to three further angles:

$$[m-1\quad m\quad m+1\quad 0] = \tfrac{1}{2}\pi - [0\quad m-1\quad m\quad m+1]$$
$$= \tfrac{1}{2}\pi - B_{m-1}B_m,$$

$$[m\quad m+1\quad 0\quad 1] = [0\quad 1\quad m\quad m+1] = B_1\,B_m,$$

$$[m+1\quad 0\quad 1\quad 2] = \tfrac{1}{2}\pi - [0\quad 1\quad 2\quad m+1] = \tfrac{1}{2}\pi - B_1\,B_2.$$

2

These $m+2$ angles can be described symmetrically as the acute angles between cyclically consecutive pairs of $m+2$ primes $1, \ldots, m, m+1, 0$ (of the spherical $(m-1)$-space), namely, of the m bounding primes of the orthoscheme, the absolute polar of B_m, and the absolute polar of B_1. Clearly, all non-consecutive pairs of these $m+2$ primes are perpendicular, and any consecutive set of m bound an orthoscheme. Schläfli considered such a cycle of $m+2$ primes in the case when $m=2n$*, and his remarks are equally valid when m is odd. When $m=3$, this is, of course, the *Pentagramma Mirificum* itself. When $m=4$, it is Wythoff's cycle of six great spheres.**

The $m+2$ primes in spherical $(m-1)$-space can be interpreted as $m+2$ concurrent primes in Euclidean m-space. The angles will not be affected by shifting two of these primes so as to destroy the concurrence; and then the removal of any one of the primes leads to a Euclidean orthoscheme, whose longest edge is perpendicular to the removed prime. In order to make the representation perfectly symmetrical, we perform an arbitrary inversion, transforming the line of the X's into a circle. Our results may now be summarized as follows :

If a cycle of $m+2$ primes (in Euclidean m-space) has the property that all non-consecutive pairs are perpendicular, then the angles between consecutive pairs are halves of the angles between consecutive arcs

$$X_{m+1}X_1,\ X_0X_2,\ X_1X_3,\ X_2X_4, \ldots,\ X_{m-1}X_{m+1},\ X_mX_0,$$

orthogonal to one circle. Every angular property of the Euclidean orthoscheme derived by omitting any one of the primes, or of the spherical orthoscheme derived by omitting any two consecutive primes, occurs as half the angle between certain arcs X_sX_q, X_pX_t $(s<p<q<t)$. Conversely, given any set of $m+2$ points on a circle, we can construct a corresponding cycle of $m+2$ primes as follows: we invert about one of the points, so as to obtain $m+1$ collinear points $X_0, X_1, \ldots, X_m$, and then draw (in Euclidean m-space) m mutually perpendicular lines

$$A_0A_1,\ A_1A_2,\ \ldots,\ A_{m-1}A_m.$$

* Schläfli, 3, *59*.

** Wythoff, 1, 582. His results concerning the volumes of the orthoschemes I, II, ..., VI had already been given by Schläfli (3, *58*) (for the general even values of m).

of lengths

$$(X_0X_1)^{\frac{1}{2}},\ (X_1X_2)^{\frac{1}{2}},\ \dots,\ (X_{m-1}X_m)^{\frac{1}{2}};$$

then $m+1$ of the primes are determined by sets of m of the $m+1$ A's, and the last is any prime perpendicular to A_0A_m.

4. *The use of symbols (s, t),*

such that $(r, s)\ (t, u) + (r, t)\ (u, s) + (r, u)\ (s, t) = 0.$

We now give an alternative treatment, to replace § 3. We define $[r\,s\,t\,u]$ to mean $A_s\,(A_rA_u)A_t$ $(u \leqq m)$ or $A_s(A_r)A_t$ $(u=m+1)$, and derive an expression for it which includes (1·2) as a particular case.

In the four dimensional orthoscheme $A_0A_1A_2A_3A_4$, consider the bounding primes *1, 2, 3*, *i.e.*, $A_0A_2A_3A_4$, $A_0A_1A_3A_4$, $A_0A_1A_2A_4$.

These pass through the edge A_0A_4, and form a trihedral angle in any prime perpendicular to A_0A_4. This trihedral angle, when regarded as a spherical triangle, has the sides

$$\begin{aligned} a &= A_2(A_0A_4)A_3 = [0\ 2\ 3\ 4],\\ b &= A_1(A_0A_4)A_2 = [0\ 1\ 2\ 4],\\ c &= A_1(A_0A_4)A_3 = [0\ 1\ 3\ 4],\end{aligned}$$

and the angles

$$\begin{aligned} A &= A_2(A_0A_1A_4)A_3 = A_2(A_1A_4)A_3 = [1\ 2\ 3\ 4],\\ B &= A_1(A_0A_3A_4)A_2 = A_1(A_0A_3)A_2 = [0\ 1\ 2\ 3],\\ C &= A_1(A_0A_2A_4)A_3 = \tfrac{1}{2}\pi.\end{aligned}$$

Hence, by the well-known formulae for the sides of a spherical triangle in terms of its angles,

$$\begin{aligned} \cos\,[0\,2\,3\,4] &= \operatorname{cosec}\,[0\,1\,2\,3]\,\cos\,[1\,2\,3\,4],\\ \cos\,[0\,1\,2\,4] &= \cos\,[0\,1\,2\,3]\,\operatorname{cosec}\,[1\,2\,3\,4],\\ \cos\,[0\,1\,3\,4] &= \cot\,[0\,1\,2\,3]\,\cot\,[1\,2\,3\,4].\end{aligned}$$

Since any five vertices of an m-dimensional orthoscheme* form a four-dimensional orthoscheme, it follows that

$$(4{\cdot}1)\quad \left\{\begin{aligned} \cos\,[r\,t\,u\,v] &= \operatorname{cosec}\,[r\,s\,t\,u]\,\cos\,[s\,t\,u\,v]\\ \cos\,[r\,s\,t\,v] &= \cos\,[r\,s\,t\,u]\,\operatorname{cosec}\,[s\,t\,u\,v]\\ \cos\,[r\,s\,u\,v] &= \cot\,[r\,s\,t\,u]\,\cot\,[s\,t\,u\,v]\end{aligned}\right\}(r<s<t<u<v).$$

* It is still immaterial whether we consider the Euclidean orthoscheme $A_0A_1\dots A_m$ or the spherical orthoscheme $B_1\dots B_m$. In the latter case, our formulae give the edges as well as the angles.

These formulae enable us to express any angle $[r\,s\,t\,u]$ in terms of others having a smaller value for $u-r$, and so ultimately in terms of the dihedral angles

$$[0\,1\,2\,3],\quad [1\,2\,3\,4],\ldots\ldots$$

We observe that they are identically satisfied in terms of symbols (s, t), such that

$$(t, s) + (s, t) = 0, \tag{4·2}$$

$$(r, s)\,(t, u) + (r, t)\,(u, s) + (r, u)\,(s, t) = 0, \tag{4·3}$$ *

$$\cos^2 [r\,s\,t\,u] = \frac{(r, s)\,(t, u)}{(r, t)\,(s, u)}, \quad \sin^2 [r\,s\,t\,u] = \frac{(r, u)\,(s, t)}{(r, t)\,(s, u)}. \tag{4·4}$$

By making the restriction

$$(t, u) + (u, s) + (s, t) = 0,$$

we could write

$$(s, t) = x_t - x_s,$$

and so deduce (1·2). For our present purposes, however, we prefer the alternative restriction

$$(s, s+1) = 1, \tag{4·5}$$

which enables us to express (s, t) as a determinant involving only the symbols

$$(0, 2), (1, 3), \ldots, (m-1, m+1);$$

namely,

$$(s, t) = \begin{vmatrix} (s, s+2) & 1 & 0 & \ldots & 0 \\ 1 & (s+1, s+3) & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & (t-3, t-1) & 1 \\ 0 & \ldots & 0 & 1 & (t-2, t) \end{vmatrix} \quad (s < t-2). \tag{4·6}$$

* Actually, (4·3) implies (4.2). For, by putting $r=s=t=u$ in (4·3), we deduce

$$(s, s) = 0.$$

Again, putting $r=s$ and $t=u$, then $r=t$ and $s=u$ and adding the results,

$$\{ (t, s) + (s, t) \}^2 = 0.$$

This is easily proved by induction, since

$$(s,\ t) = (t-2,\ t)\,(s,\ t-1) - (s,\ t-2).$$

By (4·4) and (4·5),

$$(4{\cdot}7)\quad \begin{cases} \sec^2[0\ 1\ 2\ 3] = (0,\ 2)\,(1,\ 3), \\ \sec^2[1\ 2\ 3\ 4] = (1,\ 3)\,(2,\ 4), \\ \qquad\cdots\ \cdots \\ \sec^2[m-2\quad m-1\quad m\quad m+1] = (m-2,\ m)\,(m-1,\ m+1). \end{cases}$$

Thus, if we are given the dihedral angles of the orthoscheme, we can choose one of the numbers

$$(0,\ 2),\ (1,\ 3),\ (2,\ 4),\ \ldots,\ (m-1,\ m+1)$$

arbitrarily, and deduce definite values for the rest. By (4·6), these determine all the numbers $(s,\ t)$. Thus, ultimately, (4·4) expresses any angle of the general orthoscheme in terms of its dihedral angles.

5. *Euclidean and hyperbolic space.*

On comparing Fig. i with Fig. iii, we see that the diagram for a plane triangle can be derived from that for a spherical triangle by making X_0 and X_4 recede to infinity in opposite directions. Since the representation is invariant under inversion, there is nothing special about the point at infinity; the essential fact is that X_0 and X_4 have moved into coincidence. We proceed to generalize this result by proving the following theorem:

The diagram for a Euclidean orthoscheme can be derived from the diagram for a spherical orthoscheme (of the same number of dimensions) by making X_{m+1} coincide with X_0.

In the diagram for a spherical orthoscheme, let X_{m+1} gradually approach X_0, while all the other X's (including X_0) remain fixed. Then all the angles $[r\ s\ t\ u]$ $(r < s < t < u)$ remain finite, except those which involve both X_0 and X_{m+1}, namely, $[o\ s\ t\ m+1]$, which tend to zero. But these are just the *edges* of the orthoscheme. The diagram therefore represents a spherical orthoscheme whose edges tend to zero while its angles remain finite. The limiting figure is, of course, a Euclidean orthoscheme.

E.g., m points evenly spaced along a line, with the point at infinity counted twice, represent the Euclidean orthoscheme whose vertices are the points $(1^{r-1}; 0^{m-r})$ $(r=1, 2, \ldots, m)$. Reflections in the bounding primes of this orthoscheme generate the group $[4, 3^{m-3}, 4]$, which is the complete symmetry-group of lattice points in $m-1$ dimensions. The same diagram with the point at infinity counted only once, represents a spherical orthoscheme which is the fundamental region for $[3^{m-3}, 4]$,* the complete symmetry-group of the $(m-1)$-dimensional Cartesian frame. By omitting the point at infinity altogether, we obtain a diagram for the fundamental region for $[3^{m-3}]$, the symmetric group of degree $m-1$.

Let us now continue the above process, by allowing X_{m+1} to pass into the interval between X_0 and X_1. The semi-circles on X_0X_t, X_sX_{m+1} $(s<t)$ no longer meet in a real point ; in fact,

$$\sin^2[o\ s\ t\ m+1] = \frac{X_0X_{m+1} \cdot X_sX_t}{X_0X_t \cdot X_sX_{m+1}} < 0,$$

whence $[o\ s\ t\ m+1]$ is a *hyperbolic* angle. Now the formulae (3·3) are essentially a parametric statement of the trigonometrical relations between the edges and angles of a simplex in spherical or elliptical space; and we know that such relations remain valid in hyperbolic space. Hence if, as in the present case, the formulae lead to pure-imaginary values for the *edges* of the simplex, while its *angles* remain real, we can be sure that we are dealing with a simplex in hyperbolic space.

In terms of the symbols (s, t), the space is spherical (or elliptic) if (s, t) is positive whenever $s<t$, but it becomes Euclidean or hyperbolic when $(o, m+1)$ is zero or negative, respectively. In the hyperbolic case, the pure-imaginary edge $[o\ s\ t\ m+1]$ is intimately associated with a certain real angle, namely, the *Gudermannian*

$$(5{\cdot}1) \qquad \mathrm{gd}([o\ s\ t\ m+1]/i) = \operatorname{arc\,sec}(\cos[o\ s\ t\ m+1])$$
$$= \operatorname{arc\,sec}\left(\frac{(o,s)\ (t,m+1)}{(o,t)\ (s,m+1)}\right)^{\frac{1}{2}}$$
$$= [o\ t\ s\ m+1]$$
$$= [m+1\ s\ t\ o].$$

* Coxeter and Todd, 2, *198*. The trigonometry of this orthoscheme was investigated by Schoute (1, *280-286*), especially the case when $m = 6$.

The points X which represent a hyperbolic orthoscheme are in natural (cyclic) order, save that X_0 comes immediately before X_{m+1}. Hence, we derive an associated spherical (or elliptic) orthoscheme by interchanging the names of those two points. Every edge is replaced by the corresponding Gudermannian, each of the angles

$$[o\ s\ t\ u], \quad [s\ t\ u\ m+1] \quad (o < s < t < u < m+1)$$

is replaced by the complement of the other, and the remaining angles (whose symbols involve neither 0 nor $m+1$) are unaltered. This device enables us to deduce all the properties of the hyperbolic orthoscheme from those of the more familiar spherical orthoscheme.

When $m=3$, for instance, we have Lobatschewsky's formulae* (21)–(26), which may be regarded as formulae for a spherical triangle ** with sides

$$\tfrac{1}{2}\pi - a', \quad \tfrac{1}{2}\pi - b', \quad \tfrac{1}{2}\pi - c'$$

opposite to angles

$$\tfrac{1}{2}\pi - B, \quad \tfrac{1}{2}\pi - A, \quad \tfrac{1}{2}\pi.$$

6. *Generalized Minkowskian space.*

In the generalized Minkowskian space *** $S^m T^n$, any $m+n$ perpendicular lines consist of m space-like and n time-like lines. In particular, this must hold for the perpendicular edges

$$A_0A_1, \quad A_1A_2, \quad A_{m+n-1}\,A_{m+n}$$

of any orthoscheme. We can still represent the orthoscheme by collinear points $X_0, X_1, \ldots, X_{m+n}$, satisfying (3·1). For, if A_sA_t is a time-like line, we simply make X_sX_t negative, *i.e.*, put X_t to the left of X_s. Conversely, given $m+n+1$ collinear points, numbered in any order, we can construct a corresponding orthoscheme in a space whose several dimensions are space-like or time-like according to the signs of the segments

$$X_0X_1, \quad X_1X_2, \quad \ldots, \quad X_{m+n-1}\,X_{m+n}.$$

* Lobatschewsky, 1. It is, perhaps, unfortunate that Lobatschewsky used a' to denote the *complement* of the Gudermannian of a, instead of the Gudermannian itself. His choice of notation is doubtless determined by the fact that a' is the *parallel-angle* of a.

** This can be identified with the triangle $(\gamma', a'\mu', \beta'\lambda')$ of Sommerville, 1, 65 (Fig. 36).

*** Coxeter and Todd, 1.

By varying the order in which the points are numbered, we obtain $(m+n+1)!$ orthoschemes with closely related properties. Of these, one lies in the Euclidean space S^{m+n}, and another (differing from it merely in having all lengths multiplied by i) in the Euclidean space T^{m+n}.

As before, we introduce an extra prime perpendicular to A_0A_{m+n}, so as to obtain a cycle of $m+n+2$ primes, of which all non-consecutive pairs are perpendicular. To derive a symmetrical representation, we let X_{m+n+1} denote the point at infinity in the line of the X's, and then invert this line into a circle. Each pair of X's now determines two arcs, of which we select that which does not contain X_{m+n+1}. (For the moment, we are not concerned with arcs terminated by X_{m+n+1}.) Then, of the arcs

$$X_0X_1, \quad X_1X_2, \;\ldots\;, X_{m+n-1}X_{m+n},$$

m are positive and n negative. It only remains to observe that the same partition of $m+n$ would be obtained by taking as "barrier point" X_s, instead of X_{m+n+1}, and considering the $m+n$ arcs

$$X_{s+1}X_{s+2}, \;\ldots\;, X_{m+n}X_{m+n+1}, \quad X_{m+n+1}X_0, \quad X_0X_1, \;\ldots\;, X_{s-2}X_{s-1}.$$

The same diagram, with X_{m+n+1} and X_0 specialized, can be regarded as representing the generalized hyperbolic* orthoscheme $B_1B_2\dots B_{m+n}$, which corresponds to the $(m+n)$-hedral angle at the vertex A_0 of $A_0A_1'\dots A_{m+n}$. Any vertex B_s lies in the real or ideal region of the generalized hyperbolic space according as the line A_0A_s is time-like or space-like, *i.e.*, according as the point X_s does or does not lie in the positive arc $X_{m+n+1}X_0$.

7. *Regular polytopes.*

Consider a regular polytope ** in Euclidean (or Minkowskian***) m-space. Let A_0 be a vertex, A_1 the mid-point of an edge containing this vertex, A_2 the centre of a plane face containing this edge, ... and A_m the centre of the whole polytope. Then, clearly, $A_0A_1\dots A_m$ is an orthoscheme ****. The whole polytope can, in fact, be divided into g such orthoschemes, g being the order of the symmetry-group. The edges A_sA_t of the orthoscheme are the *radii* ***** ${}_sR_t$ of the polytope; *e.g.*, A_0A_t and $A_{t-1}A_t$ are the circum- and in- radii

* By *generalized hyperbolic* space, we mean the non-Euclidean $(m+n-1)$-space whose metric is defined by an absolute quadric of signature $m-n$.

** *i.e.*, the *regular polyschema* of Schläfli, 1, *379*; 3, *103*; 4, *42-56*. See also Schoute, 2, *151-262*.

*** Coxeter, 3. **** Schläfli, 2, *295*. ***** Coxeter, 1, *338*, *350*.

of a t-dimensional element. Since the prime $A_0A_1 \ldots A_{m-2} A_m$ bisects a dihedral angle of the polytope, this dihedral angle is equal to

$$2\, A_{m-1}(A_0\, A_1 \ldots A_{m-2})A_m = 2A_{m-1}(A_{m-2})\, A_m.$$

If the polytope has the Schläfli symbol * $\{k_1, k_2, \ldots, k_{m-1}\}$, the remaining dihedral angles of the orthoscheme (apart from right angles) are **

$$(7{\cdot}1) \qquad \begin{cases} A_0\,(A_2)\,A_1 = \pi/k_1\,, \\ A_1\,(A_0\,A_3)\,A_2 = \pi/k_2\,, \\ \ldots \quad \ldots \quad \ldots \\ A_{m-2}\,(A_{m-3}\,A_m)\,A_{m-1} = \pi/k_{m-1}\,. \end{cases}$$

In accordance with our representation for a Euclidean (or Minkowskian) orthoscheme, we take $m+1$ collinear points $X_0, X_1, \ldots, X_m$, such that

$$(7{\cdot}2) \qquad X_t\, X_t = (\,{}_tR_t\,)^2\,,$$

and draw semi-circles on $X_0X_2, X_1X_3, \ldots, X_{m-2}X_m$, and vertical lines (*i.e.*, perpendiculars to the line of the X's) at X_1 and X_{m-1}. Then the first semi-circle cuts the first vertical line at angle $2\pi/k_1$, the $(p-1)$th and pth semi-circles cut each other at angle $2\pi/k_p$, and the last semi-circle cuts the other vertical line at an angle equal to the dihedral angle of the polytope. Further, since the *content* (*i.e.*, volume-analogue) of the orthoscheme is equal to $A_0A_1 . A_1A_2 \ldots A_{m-1} A_m/m\,!$, the content of the whole polytope is

$$(7{\cdot}3) \qquad \begin{aligned} S &= {}_0R_1\,.\,{}_1R_2 \ldots {}_{m-1}R_m\,.\,g/m\,! \\ &= (X_0\,X_1\,.\,X_1\,X_2 \ldots X_{m-1}\,X_m)^{\frac{1}{2}}\ g/m\,!\,. \end{aligned}$$

Conversely, given the k's, we can construct the semi-circles successively, and deduce the radii, etc. We begin by constructing a right-angled triangle $X_0\, Z_1\, X_1'$, with

$$X_0\, X_1 = (\,{}_0R_1\,)^2 = \tfrac{1}{4}$$

and $Z_1 = \pi/k_1$. This determines Y_1, the centre of the semi-circle $X_0\, Z_1\, X_2$. We then erect on $X_1\, X_2$ an isosceles triangle with angle

* We have replaced Schläfli's round brackets by curly ones, as in Coxeter, 2, 202.
** Todd, 1. (Our A's are his O's.)

$2\pi/k_2$ at its apex T_2. With centre T_2 and radius T_2X_1, we draw an arc to cut this first semi-circle in Z_2. This determines Y_2, the centre of the semi-circle $X_1Z_2X_3$. We then erect on X_2X_3 an isosceles triangle with angle $2\pi/k_3$ at its apex T_3; and so on. The examples shown in Fig. v * and Fig. vi correspond to the Euclidean polytope $\{5, 3, 3\}$ and the Minkowskian polytope $\{5, 3, 4\}$, respectively. In the latter case, X_4 lies to the left of X_0, showing that all the raddi ${}_iR_4$ are time-like. On the other hand, of the segments X_0X_1, X_1X_2, X_2X_3, X_3X_4, only the last is negative; so the space is S^3T.

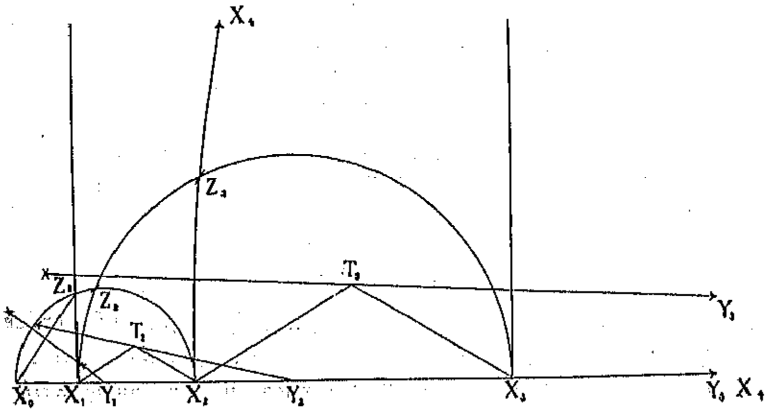

Fig. v.

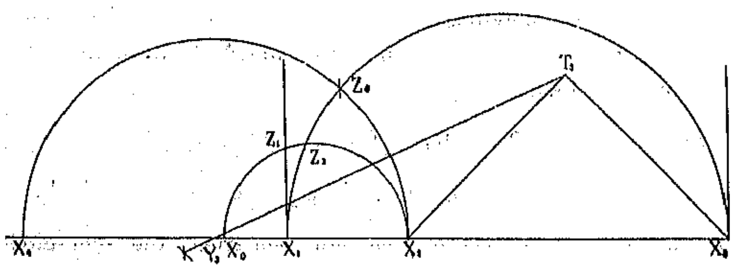

Fig. vi.

From the diagram for $\{k_1, k_2, \ldots, k_{m-1}\}$, we can deduce the diagram for the reciprocal polytope $\{k_{m-1}, \ldots, k_2, k_1\}$ by inverting (with suitable radius) about X_m, and letting X_i denote the inverse of X_{m-1-i}.

* Here the points Y_3 and X_4 are too far away to be shown. In fact, X_0X_4 is about 56 times as long as X_0X_3.

The fundamental region for the symmetry-group * of the polytope is the spherical orthoscheme cut off from $A_0 A_1 \dots A_m$ by a sphere drawn around A_m. We previously defined the spherical orthoscheme $B_1 \dots B_m$ by drawing a sphere around A_0, and reconciled the representation by calling the point at infinity X_{m+1}. In the present case, however, it is more appropriate to call the point at infinity X_{-1}, so that the vertical line at X_1 can be regarded as the semi-circle on $X_{-1} X_1$. By (4·7) and (7·1), we now have

$$(7{\cdot}4)\quad \begin{cases} \sec^2(\pi/k_1) = \sec^2[-1\ 0\ 1\ 2] = (-1,1)\,(0,2), \\ \sec^2(\pi/k_2) = \sec^2[0\ 1\ 2\ 3] = (0,2)\,(1,3), \\ \quad \dots \qquad \dots \qquad \dots \qquad \dots \\ \sec^2(\pi/k_{m+1}) = \sec^2[m-3\quad m-2\quad m-1\quad m] \\ \qquad\qquad = (m-3,\ m-1)\ (m-2,\ m). \end{cases}$$

Hence, given the k's, we can choose any one of the numbers

$$(-1,\ 1),\ (0,\ 2),\ (1,\ 3),\ \dots\ ,\ (m-2,\ m)$$

arbitrarily, and determine the rest, which then lead, by (4·6), to definite values for $(s,\ t)$ $\quad(-1 \leqq s \leqq t \leqq m)$.**

The circum-radius of a t-dimensional element is now given by

$$(7{\cdot}5)\qquad ({}_0R_t)^2 = X_0X_t = \tfrac{1}{4}X_0X_t/X_tX_1$$

$$= \tfrac{1}{4}\frac{X_{-1}X_1 \cdot X_0X_t}{X_{-1}X_t \cdot X_0X_1} = \operatorname{cosec}^2[-1\ 0\ 1\ t] = \tfrac{1}{4}\frac{(-1,\ 1)\ (0,\ t)}{(-1,\ t)\ (0,\ 1)}$$

$$= \tfrac{1}{4}(-1,\ 1)\ (0,\ t)/(-1,\ t).$$

* Goursat, 1. See also Sommerville, 2, where this spherical orthoscheme is called $C_0\, C_1 \dots C_n$. Since our work is closely related to Sommerville's, we give the following comparison of notation:

$$n = m-1,\quad R_{pq} = [-1\ p\ q\ m],\quad \phi = [-1\ 0\ 1\ 2],\quad \theta_p = [p-1\ p\ p+1\ m],$$

$$K(p,\ q) = \frac{(p-2,\ q+1)}{(p-2,\ p)\ (p-1,\ p+1)\dots(q-1,\ q+1)}.$$

** We are assuming (4·5).

Hence, by (4·2), (4·3) and (4·5),

$$(7{\cdot}6)\qquad ({}_s\mathrm{R}_t)^2 = \mathrm{X}_o\mathrm{X}_t - \mathrm{X}_o\mathrm{X}_s$$

$$= \tfrac{1}{4}(-1,1)\left(\frac{(o,t)}{(-1,t)} - \frac{(o,s)}{(-1,s)}\right) = \tfrac{1}{4}\,\frac{(-1,1)\,(s,t)}{(-1,s)\,(-1,t)}.$$

In particular,

$${}_{t-1}\mathrm{R}_t = \tfrac{1}{2}\left(\frac{(-1,1)}{(-1,t-1)\,(-1,t)}\right)^{\frac{1}{2}},$$

whence, by (7·3),

$$(7{\cdot}7)\qquad S = \frac{g}{2^m\,m!}\;\frac{(-1,1)^{\frac{1}{2}m}}{(-1,1)\,(-1,2)\ldots(-1,m-1)\;\;(-1,m)^{\frac{1}{2}}}.$$

We may note in passing, although proofs would occupy too much space, that the *simple truncation** $t_l\{k_1, k_2, \ldots\ldots, k_{m-1}\}$ has content

$$\frac{g}{2^m\,m!}\,(-1,m)^{\frac{1}{2}}\,(l-1,l+1)^{\frac{1}{2}m}\sum_{r=-1}^{l-1}\frac{(r,r)\,(r,l)^m}{(r,-1)\,(r,o)\,(r,1)\ldots(r,m)},$$

where the (r, r) in the numerator is intended to cancel (r, r) in the denominator. Similarly, the *intermediate truncation* ** $t_{l-1,l}\,\{k_1, k_2 \ldots, k_{m-1}\}$ has content

$$\frac{g}{2^m\,m!}(-1,m)^{\frac{1}{2}}\sum_{r=-1}^{l-1}\frac{(r,r)\{(r,l-1)\,(l-2,l)^{\frac{1}{2}}+(r,l)\,(l-1,\;l+1)^{\frac{1}{2}}\}^m}{(r,-1)\,(r,o)\,(r,1)\ldots(r,m)}.$$

(These results would be horribly complicated if they were expressed directly in terms of the k's.)

In conclusion, it is perhaps worth while to mention the simplest choice of values for the symbols (s, t) in the actual cases that arise in considering finite, convex polytopes. (The star polytopes—four in three dimensions, and ten in four—can be treated similarly. Minkowskian polytopes generally have finite radii, but always infinite content, because g is infinite.) For the simplex α_m,

$$(s,t) = t-s.$$

* Coxeter, 1, 354.

** Coxeter, 2, 311.

For the cross-polytope β_m,

$$(s,\,t) = t-s \quad (s \leqq t \leqq m-1), \qquad (s,\,m) = 1 \quad (s<m).$$

For the measure-polytope γ_m,

$$(s,\,t) = t-s \quad (0 \leqq s \leqq t), \qquad (-1,\,t) = 1 \quad (t \geqq -1).$$

For the 24-cell $\{3,\,4,\,3\}$,

$$(s,\,t) = t-s \quad (s \leqq t \leqq 2), \quad (s,\,3) = 1 \quad (s<3), \quad (s,\,4) = s+2 \quad (s<3).$$

For the 600-cell $\{3,\,3,\,5\}$, writing $\sigma = \frac{1}{2}(\sqrt{5}-1)$,

$$(s,\,t) = t-s \quad (s \leqq t \leqq 3),$$

$$(-1,\,4) = \sigma^6, \quad (0,\,4) = 2\sigma^4, \quad (1,\,4) = \sqrt{5}\sigma^3, \quad (2,\,4) = 2\sigma^2.$$

Finally, for the 120-cell $\{5,\,3,\,3\}$,

$$(s,\,t) = t-s \quad (0 \leqq s \leqq t),$$

$$(-1,\,1) = 2\sigma^2, \quad (-1,\,2) = \sqrt{5}\sigma^3, \quad (-1,\,3) = 2\sigma^4, \quad (-1,\,4) = \sigma^6.$$

In the last case, for example,*

$$s = \frac{g}{2^4\,4!}\,\frac{(-1,\,1)}{(-1,\,2)\,(-1,\,3)\,(-1,\,4)^{\frac{1}{2}}} = \frac{14400}{384}\,\frac{2\sigma^2}{\sqrt{5}\sigma^3 \cdot 2\sigma^4 \cdot \sigma^3}$$

$$= \frac{15}{2}\sqrt{5}\sigma^{-8} = \frac{15}{2}\sqrt{5}\tau^8,$$

where $\tau = \frac{1}{2}(\sqrt{5}+1)$.

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Stress Distribution in a Heavy Circular Disc held with its Plane Vertical by a Peg at the Centre

By

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1. In the present paper, a solution is given of the problem of the determination of the distribution of stress in a heavy circular disc, held in a vertical plane, by a peg at the centre. The problem is considered as one of generalised plane stress and the stress function χ introduced. χ is determined in two cases. In the first, the peg is assumed to be rigidly fixed to the disc, so that it exerts thrusts on the disc across the upper half of the common boundary and tensions across the lower half. In the second case, the peg is supposed to be only introduced in the circular hole, so that the peg exerts tractions on the disc, only across the upper half of the common boundary.

2. We consider the disc to be a circle of radus a, with a small concentric circular hole of radius b, which is also the boundary of the peg. The origin is taken at the centre of the disc, the axis of x, vertically upwards, and the axis of y, horizontal and in the plane of the disc. If we consider the disc to be in a state of generalised plane stress, the stress equations of equilibrium are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}-w=0, \qquad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0, \tag{1}$$

where w is the weight of the disc per unit area.

These equations are satisfied by

$$\left.\begin{aligned} X_x &= \frac{\partial^2\chi}{\partial y^2}+\frac{1}{2}wx, \qquad X_y = -\frac{\partial^2\chi}{\partial x\,\partial y}+\frac{1}{2}wy, \\ Y_y &= \frac{\partial^2\chi}{\partial x^2}-\frac{1}{2}wx, \end{aligned}\right\} \tag{2}$$

so that

$$X_x+Y_y = \nabla_1^2\chi.$$

But $$X_x+Y_y = 2(\lambda'+\mu)\Delta,$$
where λ' is the plane stress constant and is equal to $2\lambda\mu/(\lambda+2\mu)$.

The equations of equilibrium, in terms of the displacements, are

$$(\lambda'+\mu)\frac{\partial\Delta}{\partial x} + \mu\nabla^2 u - w = 0,$$

$$(\lambda'+\mu)\frac{\partial\Delta}{\partial y} + \mu\nabla^2 v = 0,$$

from which we get $$\nabla_1^2\Delta = 0,$$
so that χ satisfies the equation $$\nabla_1^4\chi = 0. \tag{3}$$

If we transform to polar co-ordinates, with the origin as the pole and the axis of x as the initial line, the tractions given by (2), are equivalent to

$$\left.\begin{aligned}\widehat{rr} &= \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\chi}{\partial r} + \frac{1}{2}wr\cos\theta,\\ \widehat{\theta\theta} &= \frac{\partial^2\chi}{\partial r^2}-\frac{1}{2}wr\cos\theta,\\ \widehat{r\theta} &= -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right)-\frac{1}{2}wr\sin\theta.\end{aligned}\right\} \tag{4}$$

3. Let us assume that

$$\chi = A(r\theta\sin\theta - \frac{\mu}{\lambda'+2\mu} r\log r\cos\theta) + Br^3\cos\theta + \frac{C}{r}\cos\theta, \tag{5}$$

which obviously satisfies the equation (3). The terms $r\theta\sin\theta$ and $r\log r\cos\theta$ give rise to many-valued displacements, and we adjust the co-efficients in such a way that the displacements become single-valued.

With the value of χ given by (5), the tractions (4) become

$$\left.\begin{aligned}\widehat{rr} &= \left[\frac{2\lambda'+3\mu}{\lambda'+2\mu}\cdot\frac{A}{r} + 2Br - \frac{2C}{r^3} + \tfrac{1}{2}wr\right]\cos\theta,\\ \widehat{r\theta} &= \left[-\frac{\mu}{\lambda'+2\mu}\cdot\frac{A}{r} + 2Br - \frac{2C}{r^3} - \tfrac{1}{2}wr\right]\sin\theta,\\ \widehat{\theta\theta} &= \left[-\frac{\mu}{\lambda'+2\mu}\cdot\frac{A}{r} + 6Br + \frac{2C}{r^3} - \tfrac{1}{2}wr\right]\cos\theta.\end{aligned}\right\} \tag{6}$$

The rim of the disc, being free from stress, we have

$$\widehat{rr}=0, \qquad \widehat{r\theta}=0, \tag{7}$$

when $r=a$.

The peg exerts a vertically upward force on the disc, equal to the weight of the disc, and we assume that the force is exerted by the surface tractions

$$\widehat{rr}=-p\cos\theta, \quad \widehat{r\theta}=0, \tag{8}$$

when $r=b$.

Hence from (6), (7) and (8), we get

$$\frac{2\lambda'+3\mu}{\lambda'+2\mu}\cdot\frac{A}{a}+2Ba-\frac{2C}{a^3}+\frac{1}{2}wa=0,$$

$$-\frac{\mu}{\lambda'+2\mu}\cdot\frac{A}{a}+2Ba-\frac{2C}{a^3}-\frac{1}{2}wa=0,$$

$$\frac{2\lambda'+3\mu}{\lambda'+2\mu}\cdot\frac{A}{b}+2Bb-\frac{2C}{b^3}+\frac{1}{2}wb=-p,$$

$$-\frac{\mu}{\lambda'+2\mu}\cdot\frac{A}{b}+2Bb-\frac{2C}{b^3}-\frac{1}{2}wb=0.$$

Solving these equations, we find that

$$\left.\begin{aligned} &A=-\tfrac{1}{2}wa^2, \qquad B=-\frac{\mu}{4(\lambda'+2\mu)}\cdot\frac{wa^2}{a^2+b^2}+\tfrac{1}{4}w, \\ &C=\frac{\mu}{4(\lambda'+2\mu)}\cdot\frac{wa^4b^2}{a^2+b^2}, \qquad p=\frac{w(a^2-b^2)}{b}. \end{aligned}\right\} \tag{9}$$

The resultant force on the boundary $r=b$, of the disc, is vertically upwards and is equal to

$$\int_{-\pi}^{\pi} p\cos^2\theta \,.\, bd\theta=\pi bp=\pi(a^2-b^2)w,$$

and this is equal to the weight of the disc.

4. It is to be noted from (8), that the traction across $r=b$, is a thrust on one half of the boundary, and a tension on the other half, so that our assumption can only be valid, if the peg is rigidly fixed to the disc. If this is not the case, it is quite probable that the disc

is in contact with the peg, only along the upper half of the circle $r=b$, so that the action of the peg on the disc, consists of thrusts, only on the upper half of the boundary $r=b$. In this case, we replace the condition (8), by the following:

When $r=b$,

$$\left.\begin{aligned} \widehat{rr} &= -p\cos\theta, && -\frac{\pi}{2} \leqq \theta \leqq \frac{\pi}{2}, \\ &= 0, && -\pi \leqq \theta < -\frac{\pi}{2}, \quad \frac{\pi}{2} < \theta \leqq \pi, \\ \text{and} \quad \widehat{r\theta} &= 0, && -\pi \leqq \theta \leqq \pi. \end{aligned}\right\} \tag{10}$$

Expanding this surface value of $\widehat{rr}$ in a Fourier's series, we can write

$$\left.\begin{aligned} \widehat{rr} &= -\frac{p}{\pi} - \tfrac{1}{2} p \cos\theta + \frac{2p}{\pi} \sum_{m=1}^{\infty} (-1)^m \frac{\cos 2m\theta}{4m^2-1}, \\ \widehat{r\theta} &= 0, \end{aligned}\right\} \tag{11}$$

when $r = b$.

We now assume

$$\chi = A_0 r^2 + B_0 \log r$$
$$+ A_1\left(r\,\theta\sin\theta - \frac{\mu}{\lambda'+2\mu} r \log r \cos\theta\right) + B_1 r^3 \cos\theta + \frac{C_1}{r}\cos\theta$$
$$+ \sum_{m=1}^{\infty} \left[A_{2m} r^{2m} + \frac{B_{2m}}{r^{2m}} + C_{2m} r^{2m+2} + \frac{D_{2m}}{r^{2m-2}} \right] \cos 2m\theta, \tag{12}$$

which is obviously a solution of (3). Substituting in (4), we have

$$\widehat{rr} = 2A_0 + \frac{B_0}{r^2} + \left[\frac{2\lambda'+3\mu}{\lambda'+2\mu}\frac{A_1}{r} + 2B_1 r - \frac{2C_1}{r^3} + \tfrac{1}{2} wr \right] \cos\theta$$
$$- \sum_{m=1}^{\infty} \left[2m(2m-1)A_{2m} r^{2m-2} + \frac{2m(2m+1)B_{2m}}{r^{2m+2}} \right.$$
$$\left. + (2m-2)(2m+1)C_{2m} r^{2m} + \frac{(2m+2)(2m-1)D_{2m}}{r^{2m}} \right] \cos 2m\theta,$$

$$\widehat{r\theta} = \left[-\frac{\mu}{\lambda'+2\mu}\frac{A_1}{r} + 2B_1 r - \frac{2C_1}{r^3} - \tfrac{1}{2} wr \right] \sin\theta$$
$$+ \sum_{m=1}^{\infty} \left[2m(2m-1)A_{2m} r^{2m-2} - \frac{2m(2m+1)B_{2m}}{r^{2m+2}} \right.$$
$$\left. + 2m(2m+1)C_{2m} r^{2m} - \frac{2m(2m-1)D_{2m}}{r^{2m}} \right] \sin 2m\theta.$$

The conditions (7) and (11), then give

$$2A_0 + \frac{B_0}{a^2} = 0, \quad 2A_0 + \frac{B}{b^2} = -\frac{p}{\pi},$$

$$\frac{2\lambda' + 3\mu}{\lambda' + 2\mu} \cdot \frac{A_1}{a} + 2B_1 a - \frac{2C_1}{a^3} + \tfrac{1}{2} wa = 0,$$

$$\frac{2\lambda' + 3\mu}{\lambda' + 2\mu} \cdot \frac{A_1}{b} + 2B_1 b - \frac{2C_1}{b^3} + \tfrac{1}{2} wb = -\tfrac{1}{2} p,$$

$$-\frac{\mu}{\lambda' + 2\mu} \cdot \frac{A_1}{a} + 2B_1 a - \frac{2C_1}{a^3} - \tfrac{1}{2} wa = 0,$$

$$-\frac{\mu}{\lambda' + 2\mu} \frac{A_1}{b} + 2B_1 b - \frac{2C_1}{b^3} - \tfrac{1}{2} wb = 0,$$

and for $m \geqq 1$,

$$2m(2m-1) A_{2m} a^{2m-2} + \frac{2m(2m+1) B_{2m}}{a^{2m+2}} + (2m-2)(2m+1) C_{2m} a^{2m} + \frac{(2m+2)(2m-1) D_{2m}}{a^{2m}} = 0,$$

$$2m(2m-1) A_{2m} b^{2m-2} + \frac{2m(2m+1) B_{2m}}{b^{2m+2}} + (2m-2)(2m+1) C_{2m} b^{2m} + \frac{(2m+2)(2m-1) D_{2m}}{b^{2m}} = -(-1)^m \frac{2p}{\pi(4m^2-1)}$$

$$2m(2m-1) A_{2m} a^{2m-2} - \frac{2m(2m+1) B_{2m}}{a^{2m+2}} + 2m(2m+1) C_{2m} a^{2m} - \frac{2m(2m-1) D_{2m}}{a^{2m}} = 0,$$

$$2m(2m-1) A_{2m} b^{2m-2} - \frac{2m(2m+1) B_{2m}}{b^{2m+2}} + 2m(2m+1) C_{2m} b^{2m} - \frac{2m(2m-1) D_{2m}}{b^{2m}} = 0.$$

Solving these equations, we have

$$A_0 = \frac{pb^2}{2\pi(a^2 - b^2)}, \qquad B_0 = -\frac{p a^2 b^2}{\pi(a^2 - b^2)},$$

$$A_1 = -\tfrac{1}{2} wa^2, \qquad B_1 = -\frac{\mu}{4(\lambda' + 2\mu)} \frac{wa^2}{a^2 + b^2} + \tfrac{1}{4} w,$$

$$C_1 = \frac{\mu}{4(\lambda'+2\mu)} \cdot \frac{wa^4b^2}{a^2+b^2}, \quad p = \frac{2w(a^2-b^2)}{b},$$

and for $m \geqq 1$,

$$4m(2m-1)\,R_{2m}\,A_{2m} = -(2m-1)\,P_{2m}\,a^2 - Q_{2m}\,a^{-4m+2},$$

$$4m(2m+1)\,R_{2m}\,B_{2m} = P_{2m}\,a^{4m+2} - (2m+1)\,Q_{2m}\,a^2,$$

$$2(2m+1)\,R_{2m}\,C_{2m} = P_{2m}, \qquad 2(2m-1)\,R_{2m}\,D_{2m} = Q_{2m},$$

where

$$P_{2m} = (-1)^m \cdot \frac{2p}{\pi(4m^2-1)}\left[(2m+1)(a^2-b^2)\,b^{-2m+2} - (a^{-4m+2}-b^{-4m+2})\,b^{2m+2}\right],$$

$$Q_{2m} = (-1)^m \cdot \frac{2p}{\pi(4m^2-1)}\left[(2m-1)(a^2-b^2)\,b^{2m+2} + (a^{4m+2}-b^{4m+2})\,b^{-2m+2}\right],$$

$$R_{2m} = (a^{4m+2}-b^{4m+2})(a^{-4m+2}-b^{-4m+2}) + (4m^2-1)(a^2-b^2)^2.$$

The force exerted by the peg on the disc, is vertically upwards and is equal to

$$\int_{-\pi/2}^{\pi/2} p\cos^2\theta \,.\, b\,d\theta = \tfrac{1}{2}\pi bp = \pi(a^2-b^2)w,$$

and this is equal to the weight of the disc.

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Some Polynomials analogous to Abel's Polynomials

By

Maurice de Duffahel.

Abel has studied a class of interesting polynomials which are defined by

$$P_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (x^n e^{-x}),$$

and can be expressed through the confluent hypergeometric function (*i.e.*, Kummer function) by the formula

$$P_n(x) = e^x \, {}_1F_1(n+1; 1; -x).$$

These polynomials arise from the generating function.

$$\frac{1}{1-h} e^{-hx/(1-h)} = \sum_{n=0}^{\infty} h^n P_n(x).$$

We propose, in this paper, to study some new polynomials, a generalization of Abel's, connected with one of the confluent hypergeometric functions of the third order.

I.

We consider the following function

$$P_n(x) = \frac{e^{x^2}}{n!} \frac{d^n}{dx^n} (x^n e^{-x^2}).$$

It is readily seen that P_n is a polynomial of degree $2n$ in x, containing only terms of even degree. The first polynomials of the series are

$$P_0 = 1,$$
$$P_1 = 1-2x^2,$$
$$P_2 = 1-5x^2+2x^4,$$
$$P_3 = 1-9x^2+8x^4-\tfrac{4}{3}x^6,$$

....

In order to connect P_n with the hypergeometric function, let us take

$$Q_n = \frac{d^n}{dx^n}(x^n e^{-x^2}),$$

so that

$$P_n = \frac{e^{x^2}}{n!} Q_n.$$

It is easy to find the differential equation (of the third order) satisfied by Q_n (which is *not* a polynomial). Let us take

$$z = x^n e^{-x^2},$$

$$y = \frac{d^n z}{dx^n} = Q_n.$$

We have

$$z' = nx^{n-1} e^{-x^2} - 2x^{n+1} e^{-x^2} = \frac{n}{x} z - 2xz,$$

so that $$xz' = nz - 2x^2 z,$$

and by successive derivation,

$$xz''' + (2-n+2x^2) z'' + 8xz' + 4z = 0.$$

Now let us differentiate the first member of this equation n times with respect to x, remembering that

$$\frac{d^n z}{dx^n} = y,$$

$$\frac{d^n z'}{dx^n} = \frac{d^{n+1} z}{dx^{n+1}} = y',$$

and so on. We obtain for y the differential equation

$$xy''' + (2x^2+2) y'' + 4(n+2) xy' + 2(n^2+3n+2) y = 0.$$

To reduce it to a known type, we take

$$u = -x^2,$$

and obtain

$$u^2 \frac{d^3y}{du^3} + \left(-u^2 - \tfrac{5}{2} u\right) \frac{d^2y}{du^2} + \left\{-(n+\tfrac{5}{2}) u + \tfrac{1}{2}\right\} \frac{dy}{du} - \frac{n^2+3n+2}{4} y = 0.$$

....

Now, the equation

$$x^2y''' - \{x-(\gamma+\delta+1)\}\, xy'' - \{x\,(\alpha+\beta+1)-\gamma\delta\}\, y' - \alpha\beta y = 0,$$

is satisfied by the confluent hypergeometric function of the third order,

$${}_2F_2\,(\alpha,\, \beta;\, \gamma,\, \delta;\, x) = \sum_{n=0}^{\infty} \frac{(\alpha,\, n)\,(\beta,\, n)}{(\gamma,\, n)\,(\delta,\, n)} \cdot \frac{x^n}{n!},$$

where the symbol $(\alpha,\, n)$ represents, as usual, the product

$$\alpha\,(\alpha+1)\,(\alpha+2)\ldots\ldots\ldots\ldots\ldots\ldots(\alpha+n-1).$$

So, by identification of the co-efficients, we have

$$y = Q_n\,(u) = Q_n\,(x) = n!\; {}_2F_2\left(\frac{n}{2}+1, \frac{n+1}{2};\, 1, \tfrac{1}{2};\, -x^2\right),$$

and we have the required value of P:

$$P_n\,(x) = e^{x^2}\; {}_2F_2\left(\frac{n}{2}+1,\, \frac{n+1}{2};\, 1,\, \tfrac{1}{2};\, -x^2\right).$$

From this formula, we can deduce a general form for the polynomial P_n.

It is known that if we consider two hypergeometric functions (of any order),

$${}_rF_s\,(a_1, a_2, \ldots a_r;\, b_1, b_2, \ldots b_s;\, x),$$

$${}_\rho F_\sigma\,(\alpha_1, \alpha_2, \ldots \alpha_\rho;\, \beta_1, \beta_2, \ldots \beta_\sigma;\, -x),$$

their product can be put in the form

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}\,\frac{(a_1, n)\ldots(a_r, n)}{(b_1, n)\ldots(b_s, n)}\; {}_{r+\rho+1}F_{s+\sigma}\,(-n,\, 1-b_1-n, \ldots \alpha_1, \ldots;\; 1-a_1-n, \ldots, \beta_1, \ldots;\, 1).$$

Now, as

$$e^{x^2} = {}_0F_0\,(x^2),$$

we have

$$P_n\,(x) = \sum_{m=0}^{\infty} \frac{x^{2m}}{m!}\; {}_3F_2\left(-m,\, \frac{n}{2}+1,\, \frac{n}{2}+\tfrac{1}{2};\, 1,\, \tfrac{1}{2};\, 1\right).$$

II.

Lagrange's well-known method will enable us to obtain for our polynomial P_n a generating function. It states that, if

$$u = x + h\phi(u),$$

then

$$f(u) = f(x) + \sum_{n=1}^{\infty} \frac{h^n}{n!} \frac{d^{n-1}}{dx^{n-1}} [\phi^n(x) f'(x)].$$

Here we have

$$f'(x) = e^{-x^2}, \quad \phi(x) = x;$$

hence

$$u = x + hu,$$

or

$$u = \frac{x}{1-h}.$$

If we differentiate the two members of the above expansion with respect to x, we have

$$\frac{d}{du} f(u) . \frac{du}{dx} = f'(x) + \Sigma \frac{h^n}{n!} \frac{d^n}{dx^n} (x^n e^{-x^2}),$$

or

$$\frac{e^{-u^2}}{1-h} = e^{-x^2} + \Sigma \frac{h^n}{n!} \frac{d^n}{dx^n} (x^n e^{-x^2}).$$

Multiplying by e^{x^2}, we have

$$\frac{e^{x^2} . e^{-x^2/(1-h)^2}}{1-h} = \sum_{n=0}^{\infty} h^n P_n(x),$$

or

$$\frac{e^{-x^2(2-h)h/(1-h)^2}}{1-h} = \sum_{n=0}^{\infty} h^n P_n(x).$$

From this generating function, we can deduce, by a well-known process of derivation with respect to x and h, the *recurrence formulae*

for P: such as

$$(n+1)\,P_{n+1}-(3n+1-x^2)\,P_n+(3n-1)\,P_{n-1}-(n-1)\,P_{n-2} = 0,$$

$$P'_n-2\,P'_{n-1}+P'_{n-2}+2x\,(2\,P_{n-1}-P_{n-2}) = 0.$$

We can also obtain an *addition formula*, as follows: let us write

$$\frac{e^{-(x^2+y^2)\,(2-h)\,h/(1-h)^2}}{1-h} = \sum_n h^n\,P_n\,(\sqrt{x^2+y^2}).$$

The first member is the product

$$(1-h)\ \sum_m h^m\,P_m\,(x)\ \sum_p h^p\,P_p\,(y)\ ;$$

so, by identification,

$$P_n\,(\sqrt{x^2+y^2}) = \sum_{m+p=n} P_m\,(x)\,P_p\,(y) - \sum_{m+p=n-1} P_m\,(x)\,P_p\,(y).$$

III.

Let us give now some integral properties. If we write

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-x^2}\,x^p\,\frac{e^{-x^2(2-h)h/(1-h)^2}}{1-h}\,dx,$$

we have, putting $x = u\,(1-h)$,

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-u^2}u^p\,(1-h)^p\,du. \tag{1}$$

But

$$\mathrm{I} = \sum h^n \int_{-\infty}^{+\infty} e^{-x^2}x^p\,P_n\,(x)\,dx. \tag{2}$$

Now, under the form (1), I contains only the powers of h from 0 to p; so, under the form (2), the same fact must occur, and the terms in h^{p+1},......are zero, so that,

$$\int_{-\infty}^{+\infty} e^{-x^2}\, x^p\, P_n(x)\, dx = 0, \qquad (p < n).$$

Thus we can write the *orthogonal property*

$$\int_{-\infty}^{+\infty} e^{-x^2}\, P_m(x)\, P_n(x)\, dx = 0, \qquad 2m < n \text{ or } 2n > m.$$

If $2m = n$, we shall obtain without difficulty

$$\int_{-\infty}^{+\infty} e^{-x^2}\, P_m(x)\, P_{2m}(x)\, dx = (-1)^m \frac{\sqrt{\pi}}{m\,!}.$$

Now, we know that if $f_m(x)$ is a polynominal of order m such as

$$\int_{-\infty}^{+\infty} e^{-x^2}\, x^p f_m(x)\, dx = 0, \qquad (p < m),$$

f_m is Hermite's polynomial

$$U_m(x) = e^{x^2} \frac{d^m}{dx^m} e^{-x^2}.$$

Here we have a polynomial P_n of order $2n$, with $p < n$: so that P_n is a sum of Hermite's polynomials, from U_n to U_{2n}.

To obtain this sum, we start from the expression of x^n as a séries of Hermite's polynomials,

$$x^n = \Sigma\, \alpha_{n-2q} U_{n-2q}(x),$$

where the α's are known constants, the sum being taken from $q=0$ to

$q=\mathrm{E}n/2$. We have

$$x^n e^{-x^2} = \Sigma\, a_{n-2q} \frac{d^{n-2q}}{dx^{n-2q}} e^{-x^2},$$

and
$$\frac{d^n}{dx^n}(x^n e^{-x^2}) = \Sigma\, a_{n-2q} \frac{d^{2n-2q}}{dx^{2n-2q}} e^{-x^2}$$

so that
$$n!\, P_n(x) = \Sigma\, a_{n-2q} \mathrm{U}_{2n-2q}(x).$$

Other formulae can be written, connecting P_n with the polynomials U. For instance, as

$$n!\, P_n(x) = e^{x^2} \frac{d^n}{dx^n}(x^n e^{-x^2})$$

$$= e^{x^2} \left\{ x^n \frac{d^n}{dx^n} e^{-x^2} + \mathrm{C}_n^1 \frac{d}{dx} x^n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} + \dots\dots \right\}$$

$$= x^n \mathrm{U}_n + \mathrm{C}_n^1 \mathrm{U}_{n-1} \frac{d}{dx} x^n + \dots\dots \quad \dots\dots,$$

and
$$\frac{d^p x^n}{dx^p} = p!\ \mathrm{C}_n^p x^{n-p},$$

we can write

$$n!\, P_n(x) = \sum_{p=0}^{n} p!\, (\mathrm{C}_n^p)^2 x^{n-p} \mathrm{U}_{n-p},$$

the symbol C_n^p standing for the number of combinations of n objects p at a time.

We have thus obtained a number of properties for our polynomials P_n. Some other properties could be written, for instance, the expression for P_n as a determinant; its connection with Abel's polynomials; its properties as a Sturm series, etc. As polynomials

connected with the hypergeometric function of the third order have scarcely been studied, all these properties are interesting, and may be extended to other more complicated polynomials of the same type.

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13

THEORY OF SKEW RECTANGULAR PENTAGONS OF HYPERBOLIC SPACE

I. Derivation of the set of associated pentagons

BY

R. C. BOSE.

Introduction.

It is well known that to any right-angled triangle on the Hyperbolic plane, there correspond four other right-angled triangles, five tri-rectangular quadrilaterals, and one rectangular pentagon, any element of each of these later figures being uniquely determined by a corresponding element of the given right-angled triangle.* These figures are shown in Fig. 1. Besides these Hyperbolic plane figures

* Nikolai Jwanwitsch Lobaschfeskij, 'Zwei Geometrische Abhandlungen' (Leipzig 1898-99), *15* and *22*. Anmerkungen Von F. Engel, *242* and *250*.

F. Engel, 'Zur nichteuklidischen Geometrie.' Leipzig. Ber. Ges. Wiss. Math. Phy. Klasse, 50, *181* (1898).

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D. M. Y. Sommerville, 'Non-Euclidean Plane Geometry,' *74*.

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S. Mukhopadhyaya, 'Geometrical Investigations on the correspondences between a right-angled triangle, a three right angled quadrilateral and a rectangular pentagon in Hyperbolic Geometry.' Bull. Cal. Math. Soc., 13, *211* (1922-23). Also Collected works, Part I, *64-68*.

M. Simon, 'Nichteuklidische Geometrie' (Teubner. Leipzig und Berlin. 1925), *48* and *54*.

E. Roeser, 'Die komplementären Figuren der nichteuklidischen Ebene.' Sitzungsberichte der Heidelberger Akademie, 2. Abhandlung, *6* (1925).

R. C. Bose, 'The theory of associated figures in Hyperbolic Geometry,' Bull. Cal. Math. Soc., 19 (1928), *109*.

there is also a spherical star pentagon (carrying five right-angled triangles) shown in Fig. 2,* whose elements are also uniquely

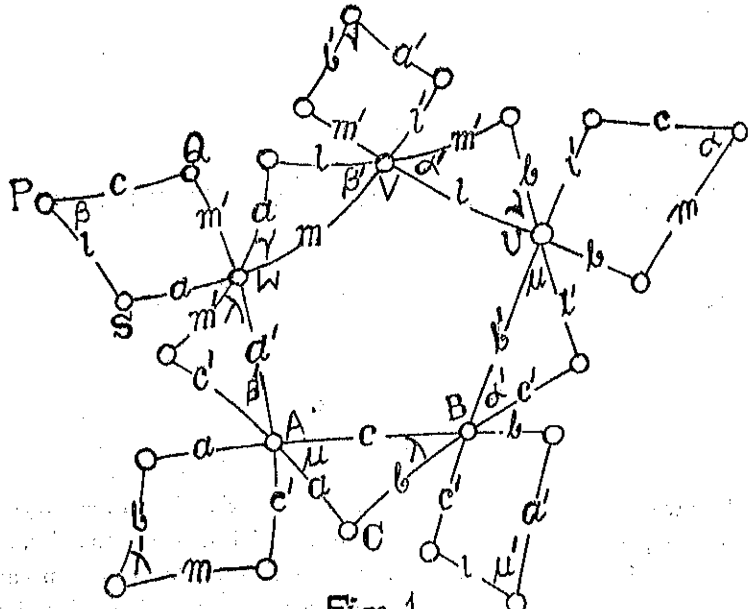

Fig. 1

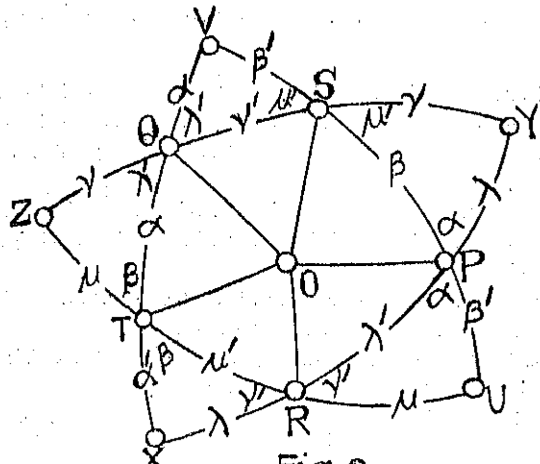

Fig 2

* In indicating the elements of the various triangles, etc., in Figs. 1 and 2, we have followed the usual notation, *viz.*, α, β, γ, λ, μ denote the angles of parallelism corresponding to the distances a, b, c, l, m respectively. α', β', γ', λ', μ' denote angles complementary to α, β, γ, λ, μ; while a', b', c', l', m' denote the distances of parallelism corresponding to the angles α', β', γ', λ', μ'.

determined by those of the given right-angled triangle.* The twelve figures taken together form a system of associated figures, the existence of any one figure ensuring the existence of the rest. The object of this paper is to show, that *the known system of associated figures, is only a degenerate case of a more general system, viz., the system of 12 associated skew rectangular pentagons, in Hyperbolic 3-space.* The geometrical inter-relations between the figures of the generalised set, and the bearing of the theory of groups on the matter, will be studied in later papers of this series.

§ I

Complex segments and skew rectangular pentagons.

1. Certain preliminary generalisations are necessary, in order to build up a general theory of associated figures which should include within itself the theory of associated figures, (depending on five elements) of the Non-Euclidean plane geometries. In the present section we first introduce the idea of the *complex segment* (and its measure) and derive some of its elementary properties. The *skew rectangular pentagon* is next defined, and it is shown that right-angled triangles, tri-rectangular quadrilaterals and rectangular pentagons of the Hyperbolic plane, together with the spherical rectangular pentagon, can be regarded as degenerate cases of skew rectangular pentagons.

2. The figure formed by two lines p and q in Hyperbolic space may be called a *complex segment* $\overline{pq}$ (Fig. 8). The complex segment $\overline{pq}$ is said to be proper, when p and q are not parallel. In this case, p and q possess a common perpendicular r, which may be called the axis of $\overline{pq}$. The complex segment $\overline{pq}$ is said to be improper when p and q are parallel.

A transformation of Hyperbolic space which is the resultant of an even number of reflections in planes, may be called a rigid motion, while the resultant of an odd number of reflections may be called a symmetric transformation.

* Nikolai Jwanwitsch Lobaschefskij, *loc. cit.*, *10.* Also 'Theory of Parallels' (translation by G. B. Halstead), *36.*

E. Roeser, 'Neue Sätze über sphärische und hyperbolische Fünfecke,' Sitzungsberichte der Heidelberger Akademie, 10 Abhandlung (1920), *13.*

D. M. Y. Sommerville, 'Non-Euclidean Geometry, *66.*

M. Simon, *loc. cit.*, *75.*

Two complex segments $\overline{pq}$ and $\overline{p_0q_0}$ may be said to be congruent when there exists a rigid motion, transforming p to p_0 and q to q_0. They may be said to be symmetric, when there exists a symmetric transformation converting p to p_0 and q to q_0.

To every complex segment we now want to assign a number which is invariant under the group of rigid motions, *i.e.*, we require all congruent complex segments to correspond to the same number and converselly, all complex segments corresponding to the same number to be congruent. This can be done as follows :—

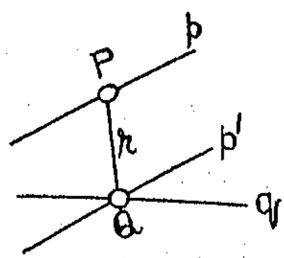

Fig. 3

Firstly, let the complex segment $\overline{pq}$ be proper, and let r be the axis of $\overline{pq}$. Let us assign an arbitrary positive sense of translation along r. We can then assign a positive sense of rotation around r, which bears to the positive sense of translation along it, the same relation, which the rotation of a right handed screw bears to the translation of the screw. A screw motion about the oriented axis r can be characterised by a complex number $d+i\phi$, where d measures the translation along r, and ϕ the rotation around r. Of course, d is a positive or a negative number, according as the translation agrees or disagrees in sense, with the assigned positive sense along r, and the same holds for ϕ. Consider now the screw motions which convert p to q. Let P and Q be the points in which the lines p and q meet r, and let p' be the line, in which the plane pr meets the plane perpendicular to r at Q (Fig. 3). Let d be the distance PQ measured positively in the assigned sense along r. Then a translation d along r brings p to p'. Next let a rotation through ϕ about r measured positively in the assigned sense, bring p' to q. Then $d+i\,\phi$ is a screw bringing p to q. If n is any integer (+ve, zero, or −ve) then $d+i(\phi+n\pi)$ would also characterise a screw bringing p to q. It is now to be remembered that the positive sense along r was arbitrarily assigned. If we had reversed the positive sense, we should have arrived at the complex numbers $-\{d+i(\phi+n\pi)\}$, as characterising the screws bringing p to q. *Thus starting with the proper complex segment $\overline{pq}$ we can arrive at the set of numbers $\pm\{d+i\,(\phi+n\pi)\}$. They will be called the set of measures of $\overline{pq}$.* (1·20).

To any improper complex segment $\overline{pq}$ *we coventionally assign the set of measures* $\pm i\,n\pi$. (1·21)

The set of measures of a complex segment $\overline{pq}$ *is independent of the order* p, q. (1·22) This is evident when $\overline{pq}$ is improper. When $\overline{pq}$ is proper, let r be as before the axis of $\overline{pq}$. Then the result follows by noting that, with an assigned positive sense along r, the screws carrying p to q are characterised by the same complex numbers as the screws carrying q to p, with the positive sense along r reversed.

Since lengths and angles are invariant under rigid motions, and a right handed system of axes is converted into a right handed system, it is clear that *the set of measures of a proper complex segment is invariant under rigid motions. The same is seen to be true for improper complex segments,* (1·23) if we note that a rigid motion converts an improper complex segment to an improper complex segment, and that any two improper complex segments have, by our convention, the same set of measures. Conversely, it is easy to see that *two complex segments with the same set of measures are congruent.* (1·230)

To assign now a unique number to every complex segment, two methods are open to us. The first method is to restrict d and ϕ so that they satisfy one of the two following relations :

$$\begin{aligned} &d > o, \quad -\pi/2 < \phi \leqq \pi/2, \\ &d = o, \quad\quad o \leqq \phi \leqq \pi/2. \end{aligned} \tag{1·24}$$

The point $d + i\phi$, if represented on the Argand diagram (Gauss plane), now occupies a region lying to the +ve side of the axis of imaginaries and bounded by the axis of imaginaries, and two lines parallel to the axis of reals at a distance $\pi/2$ on either side of it, those parts of the boundary which lie to the negative side of the axis of reals, being excluded. We can call this domain on the complex plane, the fundamental domain. It is shown in Fig. 4, as the shaded region, excluded portions of the boundary being shown by broken lines. *Among the measures of a complex segment, one and only one lies in the fundamental domain, and we can call this the principal measure of the complex segment.* (1·25)

It follows from (1·23) and (1·25), that *two congruent segments have the same principal measure and conversly.* (1·26) We shall use the notation (pq) to denote the principal measure of a complex segment. Again, it follows from (1·22) that

$$(pq) = (qp) \tag{1·27}$$

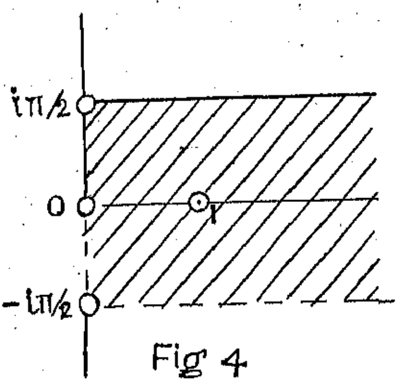

Fig 4

This will otherwise appear a little later when we establish that there exist rigid motions, interconverting p and q (*cf.* 1·30 and 1·32).

The second method would be to introduce a many-one transformation of the complex plane, which takes over each of the numbers $\pm\{d+i(\phi+n\pi)\}$ into the same number $x+iy$, while the reverse transformation takes over $(x+iy)$ to any of the numbers $\pm\{d+i(\phi+n\pi)\}$, but to no other. Such a transformation is provided by $(x+iy)=\tanh^2(d+i\phi)$. It converts what we have called the fundamental domain into the whole complex plane. Thus, if $d+i\phi$ is any measure of a complex segment, we can call $x+iy$ the transformed measure of the segment. *To every complex segment, then corresponds a unique complex number $x+iy$, which is the transformed measure of the segment, and conversely to every complex number $x+iy$, there exists a system of complex segments, all congruent to one another, whose transformed measure $x+iy$ is.** (1·28)

8. Before further developments are possible, certain elementary properties of complex segments are required.

Let $\overline{t_2 t_3}$ be a proper complex segment with axis q. Let Φ_1 and Φ'_1 be the two planes bisecting the dihedral angles between the planes qt_2 and qt_3, and let ρ be the plane bisecting perpendicularly the shortest distance between t_1 and t_2. Then the lines p_1, p_1' in

* The introduction of the i in this paper, is made only for bringing the investigations here, in line with analytical developments to follow in later papers of this series. So far as the requirements of the present paper are concerned, we might have defined the measure of a segment by the number pair (d, ϕ), where d and ϕ satisfy the restrictions (1·24). The functions considered in § 2, would then act on such number pairs.

which the plane ρ meets the plane Φ_1 and Φ_1' respectively may be defined as the right-bisectors of the complex segment $\overline{t_2 t_3}$ (Fig. 5). To distinguish between Φ_1 and Φ'_1, we may proceed as follows. If Ω_2, Ω_2' be the points at infinity on t_2 and Ω_3, Ω_3' the points at infinity on t_3, then we can take Φ_1 to be the plane, which bisects the diherdal angle between the half plane Σ_2' having the edge q and containing Ω_2, and the half plane Σ'_3 having the edge q and containing Ω_3. In the same way we can take Φ_1' to be the plane, which bisects the dihedral angle between the half-plane Σ_2' and the half-plane Σ_3'' having the edge q and containing Ω_3'.

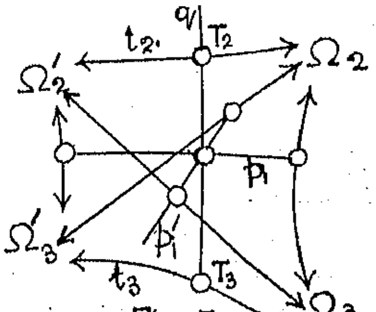

Fig. 5

Let T_2 and T_3 be the points in which t_2 and t_3 meet q, and let P_1 be the mid-point of T_2T_3. Then it is clear that the right-bisectors p_1 and p_1' pass through P_1. Let κ_2 and κ_3 be the planes perpendicular to q and passing through T_2 and T_3 respectively. A reflection in the plane ρ interchanges the planes κ_2 and κ_3, while a reflection in the plane Φ_1, interchanges the plane qt_1 with the plane qt_2 (the half-planes Σ_2' and Σ_3' being interchanged). Hence if the two reflections follow one another, the line t_2 which is the intersection of the plane κ_2 and the plane qt_2, would be interchanged with the line t_3 which is the intersection of the plane κ_3 and the plane qt_3. Since the half-plane Σ_2' is interchanged with Σ_3', the point at infinity Ω_2 on t_2 is interchanged into the point Ω_3 on t_3. The other point at infinity Ω_2' on t_2 is interchanged with the other point at infinity Ω_3' on t_3. Now the resultant of the two reflections is nothing else than a rotation through an angle π about p_1. Hence we conclude: *a rotation through an angle π, about any right bisector of a proper complex segment, interchanges the arms of the complex segment.* (1·30)

Since the rotation through an angle π about p_1 interchanges Ω_2 and Ω_3 as also Ω_2' and Ω_3', the lines $\Omega_2\Omega_3$ and $\Omega_2'\,\Omega_3'$ must meet p_1

perpendicularly. In the same way, it appears that the lines $\Omega_2\Omega_3'$ and $\Omega_2'\Omega_3$ meet p_1' perpendicularly. Since the common perpendicular to two lines is uniquely determinate, we can otherwise state the above result in the form : *Any line perpendicular to the axis q of a proper complex segment $\overline{t_2t_3}$ and also to one of the four common parallels to t_2 and t_3, must be one of the two right bisectors of the complex segment $\overline{t_2t_3}$.* (1·31)

So far we have dealt only with proper complex segments. If $\overline{t_2t_3}$ is an improper complex segment, then t_2 and t_3 are parallel, and have a common point at infinity. We may thus take Ω'_2 and Ω'_3 as coincident. In this case our previous definition of the right bisectors breaks down. We may however call the line p_1 which is perpendicular to the common parallel $\Omega_2\Omega_3$ and passes through Ω'_3, the right bisector of t_2t_3. It is then easily seen, that *a rotation through an angle π about the right bisector p_1 of an improper complex segment $\overline{t_2t_3}$ interchanges t_2 and t_3.* (1·32)

4. Any trianlge, all of whose angular points are at infinity, may be called an *asymptotic triangle.* The following property of the asymptotic triangle will be subsequently used.

Lemma. *If a line q possesses common perpendiculars p_1, p_2, p_3 with the sides $\Omega_2\Omega_3$, $\Omega_3\Omega_1$ and $\Omega_1\Omega_2$ of an asymptotic triangle $\Omega_1\Omega_2\Omega_3$ and if t_1, t_2, t_3 are lines perpendicular to q and passing through $\Omega_1, \Omega_2, \Omega_3$ respectively, then the right-bisectors of the complex segments p_1t_1, p_2t_2 and p_3t_3 coincide respectively with the right-bisectors of the complex segments* $\overline{p_2p_3}$, $\overline{p_3p_1}$ and $\overline{p_1p_2}$ (Fig. 6). (1·40)

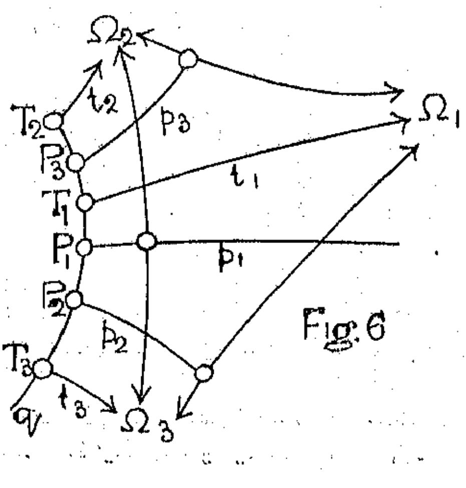

Fig. 6

Let $T_1, T_2, T_3, P_1, P_2, P_3$ denote respectively the points at which the lines $t_1, t_2, t_3, p_1, p_2, p_3$ meet q, and let $x_1, x_2, x_3, y_1, y_2, y_3$, denote the distances of these points from a fixed origin O on q (the distances being measured positively in a fixed sense along q). It follows from (1·31) that p_1, p_2, p_3 are respectively right-bisectors of the complex segments $\overline{t_2t_3}$, $\overline{t_3t_1}$ and $\overline{t_1t_2}$. Hence P_1, P_2, P_3 are the mid-points of T_2T_3, T_3T_1 and T_1T_2 respectively. Therefore

$$y_1 = \tfrac{1}{2}(x_2+x_3),$$
$$y_2 = \tfrac{1}{2}(x_3+x_1),$$
$$y_3 = \tfrac{1}{2}(x_1+x_2).$$

The distance of the mid-point of P_1T_1 from O is

$$\tfrac{1}{2}(x_1+y_1) = \tfrac{1}{4}(2x_1+x_2+x_3).$$

Also the distance of the mid-point of P_2P_3 from O is

$$\tfrac{1}{2}(y_2+y_3) = \tfrac{1}{4}(2x_1+x_2+x_3).$$

Hence *the mid-point of* P_1T_1 *coincides with the mid-point of* P_2P_3. *Denote this point by* M. (1·41)

Denote by $\Sigma_1', \Sigma_2', \Sigma_3'$ the half-planes with the edge q and containing the points $\Omega_1, \Omega_2, \Omega_3$ respectively; and by Φ_1, Φ_2, Φ_3 the planes passing through q and containing the lines p_1, p_2, p_3 respectively. A rotation through an angle π about p_1, interchanges t_2 and t_3, Ω_2 being interchanged with Ω_3. Hence the half-planes Σ_2' and Σ_3' are interchanged, while Φ_1 is unchanged. Hence they are equally inclined to Φ_1 in opposite senses. Now let χ be any half-plane with the edge q. Of the two half-planes in which Φ_1 is divided by q let Φ_1' be that which is separated from χ by Σ_2' and Σ_3'. Let $\sigma_2', \sigma_3', \phi'_1$ be the angles (lying between 0 and 2π) through which χ has to rotate in a fixed positive sense round q to arrive at Σ_2', Σ_3' and Φ_1 respectively (Fig. 7).

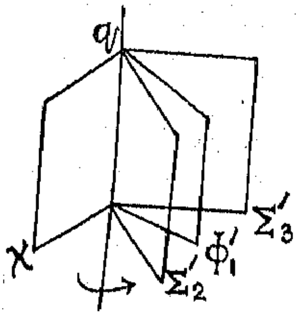

Fig. 7

Then

$$\phi_1' = \tfrac{1}{2}(\sigma_1'+\sigma_2'). \tag{1·42}$$

The line q divides Φ_2 into two half-planes, of which let Φ_2' be that which is separated from χ by Σ_1' and Σ_2'. In a similar manner, we define the half-plane Φ_3'. Let $\sigma_1', \phi'_2, \phi'_3$ be angles through which χ has to turn in the assigned positive sense round q, to arrive at $\Sigma_1', \Phi_2', \Phi_3'$ respectively. Then analogous to (1·42) we have,

$$\phi_2' = \tfrac{1}{2}(\sigma_3'+\sigma_1'). \tag{1·43}$$
$$\phi_3' = \tfrac{1}{2}(\sigma_1'+\sigma_2'). \tag{1·44}$$

The angle through which χ has to turn in arriving at the half-plane bisecting the dihedral angle between Σ_1' and Φ_1' is

$$\tfrac{1}{2}(\phi_1'+\sigma_1') = \tfrac{1}{2}(2\sigma_1'+\sigma_2'+\sigma_3'),$$

and the angle through which χ has to turn in arriving at the half-plane bisecting the dihedral angle between Φ_2' and Φ_3' is

$$\tfrac{1}{2}(\phi_2'+\phi_3') = \tfrac{1}{2}(2\sigma_1'+\sigma_2'+\sigma_3').$$

$\therefore$ Consequently the half-plane bisecting the dihedral angle between Φ_2' and Φ_3' coincides, with the half-plane bisecting the angle between Σ_1' and Φ_1'. If Σ_1 denotes the complete plane of which Σ_1' is a part, then the two planes which bisect the four dihedral angles contained by Σ_1 and Φ_1, are identical with the two planes which bisect the dihedral angles between Φ_2 and Φ_3. Taking into consideration the result (1·41), it follows that the two lines, in which these two bisecting planes meet the plane perpendicular to q through M, are the right-bisectors of both the complex segments $\overline{p_1 t_1}$ and $\overline{p_2 p_3}$.

In the same way, we can prove that the right-bisectors of the complex segments $\overline{p_2 t_2}$ and $\overline{p_3 t_3}$ coincide respectively with the right-bisectors of the complex segments $\overline{p_3 p_1}$ and $\overline{p_1 p_2}$.

5. *Given two lines p_1 and q intersecting perpendicularly at* P_1, *it is required to find a line intersecting q perpendicularly, and forming with p_1 a complex segment of given principal measure $z = d+i\,\phi$. We shall show that there are only two possible positions of the required line, which are interchangeable by rotation through an angle π about p_1.*

(1·50) The complete set of measures for the complex segment in question is $\pm\{d+i(\phi+n\pi)\}$. The line q will be the axis of the complex segment formed with p_1 by the required line.

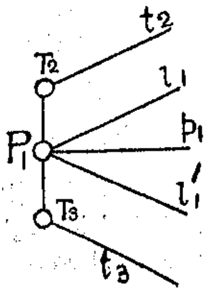

Fig. 8

Let us assign an arbitrary positive sense of translation along q. This also determines a positive sense of rotation around q as explained before. Suppose a rotation through an angle ϕ about the oriented axis q brings p_1 to l_1, and a rotation through an angle $-\phi$ brings it to l_1' (Fig. 8). Further let a translation through d along the oriented axis q bring l_1 to t_2, and a translation through $-d$ about the same axis bring l_1 to t_3. Then clearly all the screws $\{d+i(\phi+n\pi)\}$ take over p_1 to t_2, while all the screws $-\{d+i(\phi+\eta\pi)\}$ take over p_1 to t_3. If we reverse the positive directin along q, we arrive at the same lines t_2, t_3 but in

the other order. Hence t_2 and t_3 are the only positions of the required line. Let t_2 and t_3 meet q in T_2 and T_3 respectively. Now $P_1T_2 = T_3P_1 = d$. Hence P_1 is the mid-point of T_2T_3. Thus p_1 lies on the plane ρ bisecting the shortest distance T_2T_3 between t_2 and t_3. The planes qt_2 and qt_3 are identical with the planes ql and ql', and are therefore equally inclined (at an angle ϕ) to the plane qp_1. Hence the plane qp_1 bisects one pair of dihedral angles between the planes qt_2 and qt_3. Since p_1 is the intersection of the plane ρ and the plane qp_1, it is by definition, a right bisector of the complex segment $\overline{t_2t_3}$. Hence from (1·30), a rotation through an angle π about p_1 interchanges t_2 and t_3.

6. Five lines p_1, p_2, p_3, p_4, p_5 may be said to form a *skew rectangular pentagon* $p_1p_2p_3p_4p_5$, if p_1 and p_2, p_2 and p_3, p_3 and p_4, p_4 and p_5, p_5 and p_1 intersect perpendicularly. Each of these lines may be called a *side* of the pentagon. The principal measures of the complex segments $\overline{p_5p_2}$, $\overline{p_1p_3}$, $\overline{p_2p_4}$, $\overline{p_3p_5}$, $\overline{p_4p_1}$ may be said to be the *elements* of the pentagon. *If these measures are* s_1, s_2, s_3, s_4, s_5, *then the pentagon may be denoted by any one of the ten equivalent symbols* obtained from

$$\{s_1, s_2, s_3, s_4, s_5\},$$

by all possible cyclic and anticyclic permutations.

Any element of a skew rectangular pentagon may be said to correspond to that side of the skew rectangular pentagon, which forms the axis of the complex segment, of which the element in question is the measure. Thus s_3 corresponds to the side p_3, since p_3 is the axis of the complex segment $\overline{p_2p_4}$ of which the measure is s_3. *To every order of writing down the sides of the skew rectangular pentagon, we can associate one order of writing down the elements. Thus, if the elements* s_1, s_2, s_3, s_4, s_5 *correspond to the sides* p_1, p_2, p_3, p_4, p_5 *respectively, then corresponding to the order* p_3, p_2, p_1, p_5, p_4 *of writing down the sides we have the order* $\{s_3, s_2, s_1, s_5, s_4\}$ *of writing down the elements.* (1·60)

Two skew rectangular pentagons may be said to be congruent, when there exists a rigid motion by which the corresponding sides of the pentagons may be brought into coincidence. It follows from (1·26), that *the corresponding elements of two congruent pentagons are equal.* (1·61)

7. *Any two consecutive elements of a skew rectangular pentagon completely determine all the other elements.* (1·70)

Let the elements z_1 and z_5 be given. Let p_1 and p_5 be any two lines meeting perpendicularly at O (Fig. 9). Let us try to find lines p_2, p_3, p_4 such that they together with p_1 and p_5 form a skew rectangular pentagon for which

$$(p_5p_2) = z_1, \quad (p_4p_1) = z_5.$$

There are only two possible positions of the line p_2 satisfying the required conditions, and these are interchangeable by a rotation through an angle π about p_5 (*cf.* 1·50). Denote these two positions by p_2' and p_2''. Likewise there are only two possible positions of p_4, say p_4' and p_4'' which are interchangeable by a rotation through an angle π about p_1.

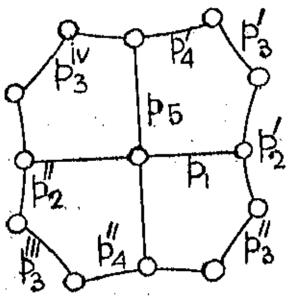

Fig. 9

Since p_3 is perpendicular to p_2 and p_4 there are four possible positions for p_3, *viz.*, p_3' the common perpendicular to p_2' and p_4', p_3'' the common perpendicular to p_2' and p_4'', p_3''' the common perpendicular to p_2'' and p_4'', and p_3^{iv} the common perpendicular to p_4' and p_2''.

Since a rotation through an angle π about p_1, interchanges p_4' and p_4'', while it leaves p_2' and p_2'' unaltered, so this rotation interchanges p_3' with p_3'' and p_3''' with p_3^{iv}. Likewise a rotation through an angle π about p_5, interchanges p_3' with p_3^{iv} and p_3'' with p_3'''.

Thus of the four skew rectangular pentagons $p_1\,p_2'\,p_3'\,p_4'\,p_5$, $p_1\,p_2'\,p_3''\,p_4''\,p_5$, $p_1\,p_2''\,p_3'''\,p_4''\,p_5$, $p_1\,p_2''\,p_3^{iv}\,p_4'\,p_5$, the first is interchanged with the second and the third with the fourth by a rotation through an angle π about p_1, while the first is interchanged with the fourth and the second with the third by a rotation through an angle π about p_5. Hence the four rectangular pentagons must be congruent. Hence elements z_1 and z_5 with which we started, unambiguously determine the other three elements z_2, z_3, z_4, *viz.*,

$$\begin{aligned} z_2 &= (p_1p_3') = (p_1p_3'') = (p_1p_3''') = (p_1p_3^{iv}), \\ z_3 &= (p_2'p_4') = (p_2'p_4'') = (p_2''p_4'') = (p_2''p_4'), \\ z_4 &= (p_3'p_5) = (p_3''p_5) = (p_3'''p_5) = (p_3^{iv}p_5). \end{aligned}$$

Corollary. *Two skew rectangular pentagons, in which two consecutive elements of the one are equal to two consecutive elements of the other, are themselves congruent.* (1·71) For, we can, by a rigid motion, bring into coincidence, the two sides of the first pentagon, which correspond to the given elements, with the two corresponding sides of the second pentagon. Then from what has been proved above, either the two pentagons will coincide, or they can be made to coincide by a further rotation through an angle π about a line.

8. We shall now show that plane right-angled triangles, tri-rectangular quadrilaterals and rectangular pentagons, as also spherical rectangular pentagons, can be regarded as special types of skew rectangular pentagons.

Let ABC be a plane right-angled triangle, right-angled at C (Fig. 1). Let AL and BM (not shown in the figure) be supposed to be drawn perpendicular to the plane of the triangle. Then the right-angled triangle in question may be regarded as a skew rectangular pentagon with sides CA, AL, AB, BM and BC. *If the plane right-angled triangle has the hypotenuse c, sides a and b, and angles λ and μ opposite to these sides respectively* (as shown in Fig. 1), *then the equivalent skew rectangular pentagon has the elements,*

$$\{a+i\pi/2,\ i\mu,\ c,\ i\lambda,\ b+i\pi/2\}. \tag{1·80}$$

Again, let PQWS be a tri-rectangular quadrilateral acute-angled at P (Fig. 1). Then it may be regarded as a skew rectangular pentagon with sides WQ, QP, PN, PS and SW, where PN (not shown in the figure) is perpendicular to the plane of the quadrilateral. *If the tri-rectangular quadrilateral has* (as shown in Fig. 1.) *the sides c, m', a, l, reckoned in order, the first and the fourth of these sides containing the acute angle β, then the equivalent skew rectangular pentagon has the elements*

$$\{m',\ c+i\pi/2,\ i\beta,\ l+i\pi/2,\ a\}. \tag{1·81}$$

Next suppose XYZUV is a spherical rectangular pentagon with sides XY $=\lambda+\pi/2$, YZ $=\gamma+\pi/2$, ZU $=\mu+\pi/2$, UV $=\pi-\beta$, VX $=\pi-\alpha$, and carrying the five spherical right-angled triangles with elements shown in Fig. 2. Let P, Q, R, S, T be the poles of the arcs VX, XY, YZ, ZU and UV respectively. If O is the centre of the sphere on which the spherical rectangular pentagon is drawn, then

it may be regarded as equivalent to a skew rectangular pentagon with sides OR, OQ, OP, OT, OS *and having the elements*

$$\{i(\pi/2-\gamma),\ i(\pi/2-\lambda),\ i\alpha, i\ \beta,\ i(\pi/2-\mu)\}. \tag{1·82}$$

Finally, it is obvious that *a plane rectangular pentagon* ABUVW (Fig. 1) *with sides a', c, b', l, m reckoned in order, can be considered to be a skew rectangular pentagon having all its sides coplanar, and possessing the elements*

$$\{a',\ c,\ b',\ l,\ m\}. \tag{1·83}$$

§2

The six fundamental functions.

1. There are four functions in ordinary Hyperbolic Geometry, *viz.*, the Π function carrying the segment a to the corresponding angle of parallelism α, the Δ function which is the inverse of the Π function and carries α to a, the complementary function which carries $a = \Delta(\alpha)$ to $a' = \Delta(\pi/2-\alpha)$ and what we may call the dual complementary function which carries α to $\alpha' = \pi/2-\alpha$. To these four funcions we should also add the identity which leaves an element unaltered. These functions fail to form a group for the simple reason, that Π and the complementary function operate only on segments (strictly lengths), while Δ and the dual complementary function operate only on angles (strictly measures of angles). If the elements of any one of the known associated figures (Figs. 1 and 2) are given, the elements of any other associated figure are connected with the elements of the given figure, just through the medium of the above mentioned functions. It is therefore necessary to investigate functions, which in the general theory of associated skew rectangular pentagons should play the corresponding role. We shall show that there exists a set of six fundamental functions 1, f, g, h, ω and ω^2 (of which the known functions of Hyperbolic Geometry are, in a certain sense, the degenerate cases) which are just suitable for our purpose. These functions operating on complex segments convert them into related complex segments (more strictly principal measures of complex segments into principal measures of related complex segments), and build a group, which is simply isomorphic with the group of rotations, converting an equilateral triangle into itself. We shall next study a certain allied transformation of Hyperbolic Line-Space, which occupies a central place in the further development of our theory.

2. *The function f.*

Def. The two points at infinity on a straight line, may, after Hilbert, be called the ends of that line.

Let u and v be any two lines forming the complex segment $\overline{uv}$ with principal measure $s = x+i\lambda$. Denote by s the common perpendicular to u and v and let Ω denote one end of u. Draw the line t perpendicular to v and passing through Ω (Fig. 10). Let $s^* = x^*+i\lambda^*$ be the principal measure of the complex segment

$\overline{st}$.

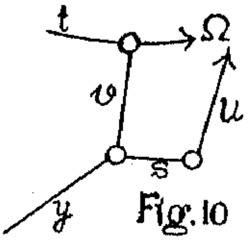

Fig. 10

Starting with any complex number s of the fundamental domain, we can thus arrive at another complex number s^* of the same domain, associated to the former through the geometrical construction given above.

In order to be sure that the number s^* is uniquely determined by s, we have only to show that we arrive at the same number, (*a*) if the line perpendicular to v is drawn to pass through the other end of u, (*b*) if the parts played by u and v in the above construction are reversed.

Of these, (*a*) is obvious for, if Ω' be the other end of u, and t' drawn perpendicular to v and passing through Ω' then from (1·31), s is a right bisector of the complex segment $\overline{tt'}$. Hence from (1·30), a rotation through an angle π about s interchanges t and t' while it leaves unaltered the line s itself. Hence the complex segment $\overline{st'}$ is congruent to the complex segment $\overline{st}$ and therefore has the same principal measure as $\overline{st}$ (*cf.* 1·26).

To prove (*b*), we have to show that Ω'' being any end of v, and t'' is the line perpendicular to u and passing through Ω'', the complex segments $\overline{st''}$ ane $\overline{st}$ are congruent. Let p be the common perpendicular to s and $\Omega\Omega''$. Then p is a right bisector of $\overline{uv}$ (*cf.* 1·31). A rotation through an angle π about p takes over u to v, Ω being carried to Ω'' (*cf.* 1·30). Hence the line t'' goes over into t, while s remains unchanged. Therefore $(st'') = (st)$.

We may therefore conceive of a function f, applicable to every complex number z of the fundamental domain, and converting it into a uniquely determinate number z^* of the same domain and state its fundamental property in the following form :—

If s and v are any two perpendicular straight lines, and u and t are lines perpendicular to s and v respectively, and parallel to one another, then

$$(st) = f(uv), \tag{2·20}$$

or denoting (st) and (uv) by z^* and z,

$$z^* = f(z). \tag{2·21}$$

From the nature of the geometrical construction connecting z and z^*, it is obvious that we also have

$$z = f(z^*),$$

i.e., th function f is involutory, and we may write

$$f^2 = 1. \tag{2·22}$$

Two special cases are of importance for us. Firstly suppose u and v are coplanar and possess a common perpendicular of length a, so that $(uv) = a$, then obviously s and t lie in the same plane and $(st) = a'$, where a' denotes as usual the length complementary to a. We may therefore write

$$f(a) = a', \text{ where } a \text{ is a real positive number.} \tag{2·23}$$

Next suppose that u and v are coplanar and include a positive acute angle α (the distance of parallelism corresponding to α being a) then t lies in the plane of u and v, while s is perpendicular to this plane. Obviously $(st) = a + i\,\pi/2$. We therefore have

$$f(i\alpha) = a + i\,\pi/2, \qquad f(a + i\,\pi/2) = i\alpha, \tag{2·24}$$

where a is a real positive number, and α lies between 0 and $\pi/2$.

3. *The function h.*

As before let u and v be any two lines forming the complex segment $\overline{uv}$ with axis s and principal measure $z = x + i\lambda$. Draw the line y perpendicular to s and v through their point of intersection (Fig. 10). Let z_0 be the principal measure of the complex segment $\overline{uy}$.

Corresponding to every complex number z of the fundamental domain we thus get a geometrically associated number z_0 of the same domain. Here it is to be remarked that we should have arrived at the same number z_0, even if the parts played by u and v are reversed, for if y' is perpendicular to s and u through their point of intersection, then a rotation through an angle $\pi/2$ about s brings u to the position y' and v to the position y so that $(vy') = (uy)$.

We may therefore conceive of a function h, applicable to every complex number z of the fundamental domain, and converting it into the uniquely determinate associated number z_0 of the same domain, and state its fundamental property in the following form :—

If s is the common perpendicular to lines u and v, and y is the line perpendicular to s and v through their point of intersection, then

$$(yu) = h\,(vu), \tag{2·30}$$

or denoting $(u\,v)$ and $(u\,y)$ by z and z_0

$$z_0 = h\,(z).$$

From the nature of the geometrical construction connecting z_0 and z it is clear that we also have

$$z = h\,(z_0),$$

so that the *function h is involutory* and we may write

$$h^2 = 1. \tag{2·31}$$

As before the two special cases in which $z = a$, and $z = i\alpha$, are of importance to us. It is readily seen that

$$h\,(a) = a + i\pi/2, \qquad h\,(a + i\pi/2) = a, \tag{2·32}$$

where a is real positive number ;

$$h\,(i\alpha) = i\,(\pi/2 - \alpha), \tag{2·33}$$

where α lies between 0 and $\pi/2$.

4. *The group of six functions generated by f and h.*

We may denote by hf the functional operation, which is the resultant of first applying the function f and then applying the function h to any complex number of the fundamental domain, and set $hf = \omega$. We shall show that $\omega^3 = 1$

Let l, m, n be three mutually perpendicular lines meeting at O. Let p be any line perpendicular to l, making $(p\ m) = z_1$ and let Ω be one of the two ends of p. Let q and r be the lines perpendicular to m and n respectively and passing through Ω (Fig. 11).

Let us set

$$(q\ l)=z_2, \quad (qn)=z_3, \quad (rm)=z_4, \quad (rl)=z_5, \quad (pn)=z_6.$$

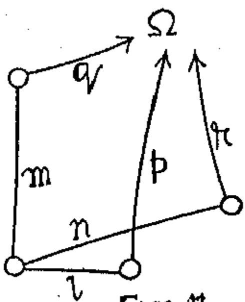

Fig 11

Then by definition

$$z_2=f\ (z_1),\ z_3=h\ (z_2),\ z_4= f\ (z_3),\ z_5=h\ (z_4),\ z_6=f\ (z_5),\ z_1=h\ (z_6).$$

Therefore $\qquad z_1= h\ f\ h\ f\ h\ f\ (z_1).$

Thus $\qquad h\ f\ h\ f\ h\ f = 1 \quad \text{or} \quad \omega^3 = 1. \qquad (2{\cdot}40)$

The results (2·22), (2·31) and (2·40) taken together show that *the functions f and h, generate a dihedral group consisting of six elements.* If we set $f\ h\ f = g$, it is easy to verify that *the group under consideration is constituted by the functions 1, f, g, h, ω, ω^2 combining according to the following multiplication table :—*

1	f	g	h	ω	ω^2
f	1	ω^2	ω	h	g
g	ω	1	ω^2	f	h
h	ω^2	ω	1	g	f
ω^2	h	f	g	1	ω
ω	g	h	f	ω^2	$1.$

(2·41)

For example to find the value of gh, we find that element in the column headed by g, which lies in the row beginning with h. Thus $gh = \omega$.

If a be any positive length, α the angle of parallelism corresponding to a, and a' the length complementary to a, it has already been seen (*cf.* 2·23 and 2·32)

$$f\ (a) = a', \quad h\ (a) = a + i\ \pi/2.$$

Further $\quad \omega^2\ (a) = f\ h\ (a) = f\ (a + i\ \pi/2) = i\ \alpha\ ,$ $\quad$ from (2·24)

$$\omega\ (a) = h\ f\ (a) = h\ (a') \qquad = a' + i\ \pi/2,$$

$$g\ (a) = \omega^2\ f\ (a) = \omega^2\ (a') \qquad = i\ (\pi/2 - \alpha).$$

Summing up we have:

$$\left.\begin{array}{c} f\ (a) = a', \quad g\ (a) = i\ (\pi/2 - \alpha), \quad h\ (a) = a + i\ \pi/2, \\ \omega\ (a) = a' + i\ \pi/2, \quad \omega^2\ (a) = i\ \alpha. \end{array}\right\} \quad (2·42)$$

5. *The B-transformation.*

Let y be any fixed line in Hyperbolic Space, and v any arbitrary line with ends Ω_1 and Ω_2. We can find two lines v_1^* and v_2^* which have one end common with v (*i.e.*, are parallel to v) while the lines joining their other ends to the other end of v intersect y perpendicularly (Fig. 12). Then the lines v_1^*, and v_2^* may be said to be the B-transforms of v, with respect to y, and the transformation of Hyperbolic Line-space which carries over every line like v into the geometrically related lines, v_1^* and v_2^* may be called the B-transformation with respect to y. This transformation institutes a (1, 2) correspondence between lines of Hyperbolic space.

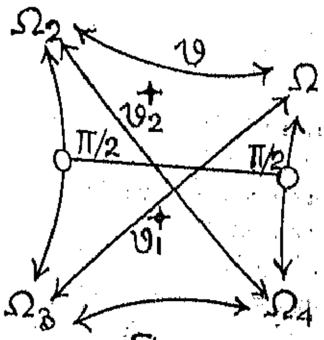

Fig. 12

It is clear that *if* v_1^* *is a B-transform* of v w. r. t. y then v_1 *is also a B-transform of* v_1^* w. r. t. y. (2·50) If we denote the ends of the lines v_1^*, v_2^* as in Fig. 12, then it is evident that v_1^*, v_2^* are also the B-transforms of the line $\Omega_3\ \Omega_4$ w. r. t. y. Thus *any two lines which can be interchanged with one another by a rotation through an angle* π *about* y *have the same two B-transforms* w. r. t. y. (2·51)

6. **Theorem.** *If the lines v and v^* are B-transforms of each other w. r. t. a line y then the common perpendiculars w and w^* which y possesses with v and v^* meet perpendicularly at a point of y and $(v^*y)=g\,(vy)$* (Fig. 13). (2·60)

Let Ω_1, Ω_2 be the two ends of v and Ω_1, Ω_3 the two ends of v^*, the line $\Omega_2\Omega_3$ being perpendicular to y. Let w be the common perpendicular to v and y meeting the latter at the point A and let w^* be drawn perpendicular to y and w through A. Now Ω_1 goes over into

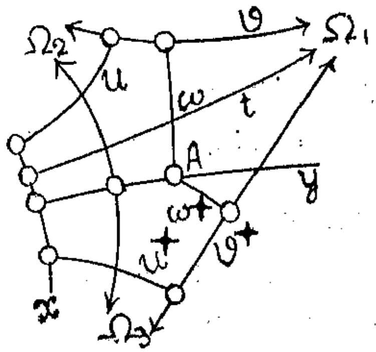

Fig 13

Ω_2 by a rotation through an angle π about w and Ω_2 goes over into Ω_3 by a rotation through an angle π about y. Since two rotations through an angle π, about two lines intersecting perpendicularly, are equivalent to a rotation through an angle π, about the line perpendicular to both at their point of their intersection, it is clear that a rotation through an angle π about w^* carries Ω_1 to Ω_3. Hence w^* meets the line $\Omega_1\Omega_3$ perpendicularly. This proves the first part of the theorem.

Again from the definition of functions f and h, it is obvious that

$$(v^*y) = h(v^*w), \quad (v^*w) = f(vw^*), \quad (vw^*) = h(vy).$$

Therefore $\quad (v^*y) = h\,f\,h\,(vy) = h\,\omega^2(vy) = g\,(vy) \quad$ (cf. 2·41)

7. **Theorem.** *If the lines v and v^* are B-transforms of each other* w. r. t. *the line y and x is any line perpendicular to y and possessing common perpendiculars u and u^* with v and v^* respectively* (Fig. 13), *then*

$$(u^*y) = f(vx), \qquad (v^*x) = f(uy). \qquad (2·70)$$

As before let $\Omega_1 \, \Omega_2$ be the two ends of v, and Ω_1, Ω_3 the two ends of v^*, the line $\Omega_2 \, \Omega_3$ being perpendicular to y. Then from (1·40) it follows that the right-bisectors of the complex segment $\overline{ty}$ coincide with those of the complex segment uu^*. A rotation through an angle π about one of these right-bisectors interchanges y with t and u with u^* (*cf.* 1·30), and therefore $(u^*y)=(tu)$ and $(uy)=(tu^*)$. But from the definition of the function f, $(tu)=f(vx)$ and $(v^*x)=f(tu^*)$. Hence $(u^*y)=f(vx)$ and $(v^*x)=f(uy)$.

§ 3.

The twelve associated skew rectangular pentagons.

1. We are now in a position to show, that corresponding to any skew rectangular pentagon with elements $\{s_1, s_2, s_3, s_4, s_5\}$, there exist eleven other skew rectangular pentagons, such that every element of each of these later pentagons is uniquely determined by a corresponding element of the original pent agon, through the medium of the six fundamental functions studied in the previous section. If in partcular, the imaginary part of the complex numbers s_1, s_2, s_3, s_4, s_5 is made to vanish, the original skew rectangular pentagon becomes a plane rectangular pentagon, while the eleven other associated skew rectangular pentagons, reduce to the eleven figures (five right-angled triangles, five tri-rectangular quadrilaterals, and one spherical rectangular pentagon) known to be associated with the plane rectangular pentagon.

2. **Theorem.** *If $x\ u\ v\ w\ y$ and $x\ u^*\ v^*\ w^*\ y$ be two skew rectangular pentagons having sides x and y common, while the sides v and v^* are B-transforms of one another* w. r. t. y (Fig. 14), *then*

$$(y\ u^*) = f\ (x\ v), \quad (x\ v^*) = f\ (yu), \quad (u^*\ w^*) = h\ (u\ w),$$

$$(v^*y) = g\ (vy), \quad (w^*x) = h(wx). \tag{3·20}$$

Now w and w^* are the common perpendiculars y possesses with v and v^* respectively. As v and v^* are B-transforms w. r. t. y we can see from (2·60) that w and w^* must meet perpendicularly at some point A of y and

$$(v^*y) = g\ (vy). \tag{3·21}$$

Again as the line x stands perpendicular to y and possesses common perpendiculars u and u^* with v and v^* respectively, we have from (2·70),

$$(yu^*) = f\,(xv),$$

$$(xv^*) = f(yu). \quad (3·22)$$

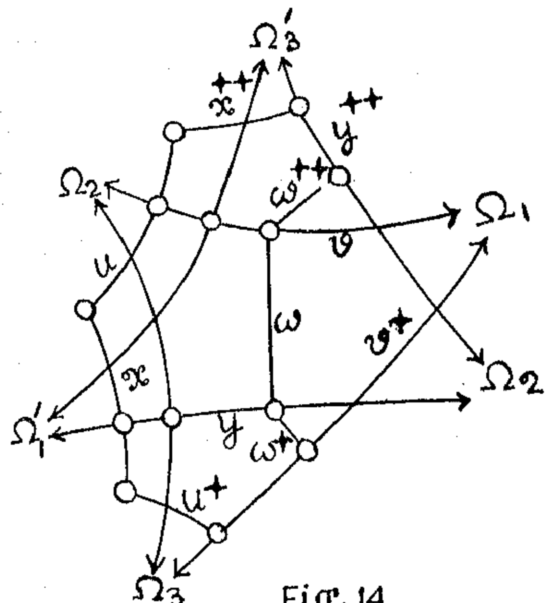

Fig. 14

Again y is the axis of the complex segment xw and w^* is the perpendicular to w and y through their point of intersection A. Hence from the definition of the function h,

$$(w^*x) = h\,(wx). \tag{3·23}$$

Again let y^{**} be a B-transform of y W. R. T. v, and let it possess common perpendiculars x^{**} and w^{**} with u and v respectively. Then the skew rectangular pentagon $x^{**}\,u\,v\;\;w^{**}\;\;y^{**}$ stands exactly in the same relation to the skew rectangular pentagon $x\,u\,v\,w\,y$, as does the skew rectangular pentagon $x\,u^*\,v^*\,w^*\,y$ (Fig. 14). Hence analogous to the results (3·22) and (3·23) we now obtain the results

$$(v\,x^{**}) = f\,(uy),\;\; (u\,y^{**}) = f\,(vx), \tag{3·24}$$

$$(u\,w^{**}) \;=\; h\,(uw). \tag{3·25}$$

Results (3·22) and (3·24) taken together show that

$$(yu^*) = (yu^{**}) \qquad \text{and} \qquad (xv^*) = (x^{**}v),$$

so that two consecutive elements of the skew rectangular pentagon $xu^*v^*w^*y$ are respectively congruent to two consecutive elements of the skew rectangular pentagon $x^{**}\,u\,v\,w^{**}\,y^{**}$. Consequently the two skew rectangular pentagons are congruent (*cf.* 1,71) and therefore

$$(u^*\,w^*) = (u\,w^{**}),$$

or from (3·25)

$$(u^* w^*) = h\ (uw), \tag{3·26}$$

Results (3·21), (3·22), (3·23) and (3·26) taken together constitute a proof of the theorem in question.

Corollary. *If the lines v and v^* are* B-*transforms of each other* w. r. t. *the line y and possess common perpendiculars w and w^* with y, and common perpendiculars u and u^* with any line x perpendicular to y, then*

$$(u^* w^*) = h\ (uw), \quad (w^* x) = h(wx). \tag{3·27}$$

3. The theorem proved in the previous paragraph, at once leads to the following geometrical construction, for deriving from a given skew rectangular pentagon, another associated rectangular pentagon:—

Let x, u, v, w, y be the sides of the given rectangular pentagon. Take two consecutive sides, x, y of the given pentagon, and find a B-transform of the side v opposite to these, with respect to the side y. Let v^* be this B-transform. Then the skew rectangular pentagon which has for three of its sides the lines x, y and v^*, is the derived skew rectangular pentagon. The other two sides are of course the common perpendiculars u^* and w^*, which x and y respectively possess with v^* (Fig. 14). (3·30)

Let $\{z_1, z_2, z_3, z_4, z_5\}$ be the elements of the original pentagon, the order of writing the elements corresponding to the order x, u, v, w, y of writing the sides of the pentagon. From theorem (3·20), it follows at once that

$$(yu^*) = f(xv) = f(z_2), \quad (xv^*) = f(yu) = f(z_1),$$

$$(u^* w^*) = h(uw) = h(z_3), \quad (v^* y) = g(vy) = g(z_4),$$

$$(w^* x) = h\ (wx) = h\ (z_5). \tag{3·31}$$

The geometrical construction (3·30), may be called the B-construction. It is clear that there are exactly ten ways of performing the B-construction on a given skew rectangular pentagon, each way corresponding to one order of writing down the sides (and therefore to one order of writing down the elements) of the given skew rectangular pentagon. The construction (3·30) may be regarded as corresponding to the order $x\ u\ v\ w\ y$ of writing down the sides, or the

order $\{z_1, z_2, z_2, z_4, z_5\}$ of writing down the elements of the original skew rectangular pentagon.

Hence we can write :—

If we perform on a given skew rectangular pentagon (P_0), *with elements* $\{z_1, z_2, z_3, z_4, z_5\}$, *the B-construction corresponding to the given order of writing down its elements ; we get an associated skew rectangular pentagon* $(P_4{}^*)$ *with elements.*

$$\{f(z_2),\ f(z_1),\ h(z_3),\ g(z_4),\ h(z_5)\}. \tag{3·32}$$

In the same way by performing on (P_0) B-contructions *corresponding to the orders* $\{z_2, z_3, z_4, z_5, z_1\}$, $\{z_3, z_4, z_5, z_1, z_2\}$, $\{z_4, z_5, z_3, z_1, z_2\}$ $\{z_5, z_1, z_2, z_3, z_4\}$ *we get the four associated skew rectangular pentagons* $(P_5{}^*)$, $(P_1{}^*)$, $(P_2{}^*)$, $(P_3{}^*)$ *with elements,*

$$\{f(z_3),\ f(z_2),\ h(z_4),\ g(z_5),\ h(z_1)\},$$

$$\{f(z_4),\ f(z_3),\ h(z_5),\ g(z_1),\ h(z_2)\},$$

$$\{f(z_5),\ f(z_4),\ h(z_1),\ g(z_2),\ h(z_3)\},$$

$$\{f(z_1),\ f(z_5),\ h(z_2),\ g(z_3),\ h(z_4)\}. \tag{3·33}$$

The remaining five orders of writing down the elements of (P_0) lead to the associated skew rectangular pentagons, already derived (the elements now coming in a different order). *From a given skew rectangular pentagon, we can thus derive through the* B-*construction exactly five associated skew rectangular pentagons.* (3·331)

If we perform on $(P_4{}^*)$ *the B-construction corresponding to the order of writing down in elements shown in* (3·32), *we get back to* (P_0). (3·332) *But corresponding to the order,*

$$\{f(z_1),\ h(z_3),\ g(z_4),\ h(z_5),\ f(z_2)\},$$

of writing down its sides, we get the new rectangular pentagon (P_1) *with elements,*

$$\{fh(z_3),\ ff(z_1),\ hg(z_4),\ gh(z_5),\ hf(z_2)\},$$

which from (2·41) *can be written as*

$$\{\omega^2(z_3),\ z_1,\ \omega^2(z_4),\ \omega(z_5),\ \omega(z_2)\}. \tag{3·34}$$

In the same way *corresponding to other orders of writing down the elements of* $(P_4{}^*)$ *we can derive from it the associated skew rectangular pentagons* (P_5), (P_3), (P_2) *with elements*

$$
\begin{array}{lllll}
\{\omega(z_4), & \omega^2(z_3), & z_5, & \omega^2(z_2), & \omega(z_1)\}, \\
\{\omega^2(z_5), & \omega(z_4), & \omega(z_2), & \omega^2(z_1), & z_3\}, \\
\{\ z_2, & \omega^2(z_5), & \omega(z_1), & \omega(z_3), & \omega^2(z_4)\}.
\end{array}
\qquad (3{\cdot}35)
$$

Thus each of the five skew rectangular pentagons $(P_4{}^*)$, $(P_5{}^*)$, $(P_1{}^*)$, $(P_2{}^*)$, $(P_3{}^*)$ leads (through the B-construction) to the original pentagon (P_0) and four new skew rectangular pentagons. But it is found that of the 20 new pentagons only 5 are distinct, each being repeated four times. *They are the skew rectangular pentagons* (P_1), (P_5), (P_3), (P_2) *whose elements are already given, and the skew rectangular pentagon* (P_4) *with elements*

$$
\{\omega^2(z_1), \quad z_4, \quad \omega^2(z_2), \quad \omega(z_3), \quad \omega(z_5)\}. \qquad (3{\cdot}36)
$$

We have up till now arrived at exactly 10 skew rectangular pentagons besides (P_0). *By performing the B-construction in all possible ways on* (P_1), (P_2), (P_3), (P_4) (P_5) *we get only one new skew rectangular pentagon* $(P_0{}^*)$ *with elements*

$$
\{g(z_2), \quad g(z_5), \quad g(z_3), \quad g(z_1), \quad g(z_4)\}. \qquad (3{\cdot}37)
$$

By performing the B-construction on $(P_0{}^*)$ we cannot get any new skew rectangular pentagon.

We can now state the main theorem of our theory in the following form :—

Theorem. *The 12 skew rectangular pentagons,* (P_0), (P_1), (P_2), (P_3), (P_4), (P_5), $(P_0{}^*)$, $(P_1{}^*)$, $(P_2{}^*)$, $(P_3{}^*)$, $(P_4{}^*)$, $(P_5{}^*)$ *whose elements are shown in the scheme below form a system of associated figures, i.e., the existence of any one figure, ensures the existence of the rest.*

$$
\begin{array}{llllll}
(P_0)-\{ & z_1, & z_2, & z_3, & z_4, & z_5\ \}, \\
(P_1)-\{ & \omega(z_5), & \omega^2(z_4), & z_1, & \omega^2(z_3), & \omega(z_2)\}, \\
(P_2)-\{ & \omega(z_1), & \omega^2(z_5), & z_2, & \omega^2(z_4), & \omega(z_3)\}, \\
(P_3)-\{ & \omega(z_2), & \omega^2(z_1,) & z_3, & \omega^2(z_5), & \omega(z_4)\}, \\
(P_4)-\{ & \omega(z_3), & \omega^2(z_2), & z_4, & \omega^2(z_1), & \omega(z_5)\}, \\
(P_5)-\{ & \omega(z_4), & {}^2(z_3), & z_5 & {}^2(z_2), & \omega(z_1)\},
\end{array}
$$

$$
\begin{array}{llllll}
(\mathrm{P}_0{}^*)-\{g(z_1), & g(z_3), & g(z_5), & g(z_2), & g(z_4)\}, & \\
(\mathrm{P}_1{}^*)-\{f(z_3), & h(z_5), & g(z_1), & h(z_2), & f(z_4)\}, & \\
(\mathrm{P}_2{}^*)-\{f(z_4), & h(z_1), & g(z_2), & h(z_3), & f(z_5)\}, & \\
(\mathrm{P}_3{}^*)-\{f(z_5), & h(z_2), & g(z_3), & h(z_4), & f(z_1)\}, & \\
(\mathrm{P}_4{}^*)-\{f(z_1), & h(z_3), & g(z_4), & h(z_5), & f(z_2)\}, & \\
(\mathrm{P}_5{}^*)-\{f(z_2), & h(z_4), & g(z_5), & h(z_1), & f(z_3)\}. & (3{\cdot}38)
\end{array}
$$

Corollary (1). From the way in which the system of 12 associated skew rectangular pentagons is derived, it is clear, that *the system is closed with respect to the B-construction*; (3·381) *i.e.*, by performing a B-construction on any member of the system (corresponding to any order of writing down its elements) we shall arrive at another member of the system. It should be noticed that *in the scheme* (3·38), *two skew rectangular pentagons* (P_i) *and* $(\mathrm{P}_j{}^*)$ *are derivable from one another through the B-construction, when and only when,* $i \neq j$. (P_i) *and* (P_j) *are never connected by a* B-*construction, and the same holds for* $(\mathrm{P}_i{}^*)$ *and* $(\mathrm{P}_j{}^*)$.. (3·382)

Corollary (2). *Any two elements of a skew-rectangular pentagon, uniquely determine the other three elements.*

This has been already proved (*cf.* 1·70), when the two given elements are consecutive elements. It remains to consider the case when the two given elements are non-consecutive.

Let $\{z_1, z_2, z_3, z_4, z_5\}$ be the elements of one skew rectangular pentagon and $\{z_1{}^*, z_2{}^*, z_3{}^*, z_4{}^*, z_5{}^*\}$ of another skew rectangular pentagon where $z_2=z_2{}^*$, $z_4=z_4{}^*$.

Corresponding to these there exist two skew rectangular pentagons with elements $\{f(z_3), h(z_5), g(z_1), h(z_2), f(z_4)\}$ and $\{f(z_3{}^*), h(z_5{}^*), g(z_1{}^*), h(z_2{}^*), f(z_4{}^*)\}$. But $h(z_2)=h(z_2{}^*)$ and $f(z_4)=f(z_4{}^*)$ since $z_2=z_2{}^*$, $z_4=z_4{}^*$. Hence the two newly arrived at skew rectangular pentagons have two consecutive elements congruent and are therefore themselves congruent (*cf.* 1·81). Consequently $f(z_3)=f(z_3{}^*)$, $h(z_5)=h(z_5{}^*)$, $g(z_1)=g(z_1{}^*)$. Therefore $z_3=z_3{}^*$, $z_5=z_5{}^*$, $z_1=z_1{}^*$. This proves our corollary.

4. The plane rectangular pentagon ABUVW (Fig. 1) whose sides reckoned in order are a', c, b', l, m can be identified with the skew rectangular pentagon (P_0) with elements $\{z_1, z_2, z_3, z_4, z_5\}$ by setting

$$z_1=a', \quad z_2=c, \quad z_3=b', \quad z_4=l, \quad z_5=m. \qquad (3{\cdot}40)$$

Using the results (2·42) it is easy to write what the elements of the 12 skew rectangular pentagons of the scheme (3·38) become in this special case. They are shown below in the following scheme:

Name. *Elements under the special assumption* (3·40).

$$
\begin{array}{ll}
(P_0) & \left\{ a', \quad c, \quad b', \quad l, \quad m, \right\}, \\[2ex]
(P_1) & \left\{ m' + \frac{i\pi}{2}, \quad i\lambda, \quad a', \quad i\left(\frac{\pi}{2}-\beta\right), \quad c' + \frac{i\pi}{2} \right\}, \\[2ex]
(P_2) & \left\{ a + \frac{i\pi}{2}, \quad i\mu, \quad c, \quad i\lambda, \quad b + \frac{i\pi}{2} \right\}, \\[2ex]
(P_3) & \left\{ c' + \frac{i\pi}{2}, \quad i\left(\frac{\pi}{2}-a\right), \quad b', \quad i\mu, \quad l' + \frac{i\pi}{2} \right\}, \\[2ex]
(P_4) & \left\{ b + \frac{i\pi}{2}, \quad i\gamma, \quad l, \quad i\left(\frac{\pi}{2}-a\right), \quad m' + \frac{i\pi}{2} \right\}, \\[2ex]
(P_5) & \left\{ l' + \frac{i\pi}{2}, \quad i\left(\frac{\pi}{2}-\beta\right), \quad m, \quad i\gamma, \quad a + \frac{i\pi}{2} \right\}, \\[2ex]
(P_0^*) & \left\{ i\left(\frac{\pi}{2}-\gamma\right), \quad i\left(\frac{\pi}{2}-\lambda\right), \quad ia, \quad i\beta, \quad i\left(\frac{\pi}{2}-\mu\right) \right\}, \\[2ex]
(P_1^*) & \left\{ b, \quad m + \frac{i\pi}{2}, \quad ia, \quad c + \frac{i\pi}{2}, \quad l' \right\}, \\[2ex]
(P_2^*) & \left\{ l', \quad a' + \frac{i\pi}{2}, \quad i\left(\frac{\pi}{2}-\gamma\right), \quad b' + \frac{i\pi}{2}, \quad m' \right\}, \\[2ex]
(P_3^*) & \left\{ m', \quad c + \frac{i\pi}{2}, \quad i\beta, \quad l + \frac{i\pi}{2}, \quad a \right\}, \\[2ex]
(P_4^*) & \left\{ a, \quad b' + \frac{i\pi}{2}, \quad i\left(\frac{\pi}{2}-\lambda\right), \quad m + \frac{i\pi}{2}, \quad c' \right\}, \\[2ex]
(P_5^*) & \left\{ c', \quad l + \frac{i\pi}{2}, \quad i\left(\frac{\pi}{2}-\mu\right), \quad a' + \frac{i\pi}{2}, \quad b \right\},
\end{array}
\tag{3.41}
$$

Now the result (1.30) shows that the existence of the specialized skew rectangular pentagon (P_2) is equivalent to the existence of a plane right-angled triangle with hypotenuse c, sides a and b and

angles λ and μ opposite to these sides respectively. This is however the triangle ABC of Fig. 1. Similarly the specialised pentagons (P_1), (P_3), (P_4), (P_5) of scheme (3·41) give rise to the other four right-angled triangles of Fig. 1.

Again the result (1·81) shows that the existence of the specialized skew rectangular pentagon $(P_3{}^*)$ is equivalent to the existence of a plane tri-rectangular quadrilateral with sides c, m', a, l taken in order, the first and the fourth sides containing the acute angle β. This is however the tri-rectangular quadrilateral PQWS of Fig. 1. In the same way the specialized skew rectangular pentagons $(P_1{}^*)$, $(P_2{}^*)$, $(P_4{}^*)$, $(P_5{}^*)$ of the scheme (3·41) give rise to the other tri-rectangular quadrilaterals of Fig 1.

Finally the result (1·82) shows that the existence of the specialised skew rectangular pentagon $(P_0{}^*)$, is equivalent to the existence of a spherical rectangular pentagon whose sides reckoned in order are $\frac{\pi}{2}+\gamma$, $\frac{\pi}{2}+\lambda$, $\pi-\alpha$, $\pi-\beta$, $\frac{\pi}{2}+\mu$. This however is the spherical rectangular pentagon, which is associated with the plane rectangular pentagon ABUVW of Fig. 1, and whose elements are shown in Fig. 2.

We have thus shown, that *the twelve associated skew rectangular pentagons, whose elements are shown in scheme* (3·38), *can as a special case degenerate to the known system of associated figures, viz., a plane rectangular pentagon, five plane right-angled triangles and five plane tri-rectangular quadrilaterals shown in Fig. 1 and one spherical rectangular pentagon shown in Fig 2.* (3·42) The point of interest however is that *whereas the figures of the known system are of diverse kinds, the figures of the generalised system are all of like nature. This introduces an order and symmetry in the study of the geometrical inter-relations between the figures of the generalised system, which is completely unattainable for the known specialised system.* This will be brought out clearly in subsequent papers of this series, where also the bearing of the group theory on the matter, will be discussed.

In conclusion, my best thanks are due to Professor Dr. S. Mukhopadhyaya for the keen interest taken by him in this investigation, and to Professor Dr. F. Levi for the many helpful suggestions given by him.

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14

The Effect of Finite Breadth of Hammer Striking a Pianoforte String.

By

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The problem of the pianoforte string has been studied by Helmholtz, Delemer, Lamb, Kaufmann and Das and more recently by Kar-Ghosh. [1] It is shown by Ghosh [2] that the theory they have developed in a different way, is perfectly general, and that all the previous theories are only special cases. It was Lord Rayleigh [3] who pointed out that '*some allowance must be made for the finite breadth of the hammer*' striking a pianoforte string. However, the effect of the finite length of contact has been neglected in the general theory of Kar-Ghosh.

In the present paper, an attempt has been made to extend the theory of Kar-Ghosh to the case when the length of contact of the hammer with the string cannot be neglected.

We suppose that the hammer has its plane surface facing the string and the portion of the string in contact with the hammer moves as a whole parallel to itself. Thus the length of contact remains constant till the hammer leaves the string. It is evident that the displacements at the two ends of contact must be same, and the resultant of the tensions acting at both ends must balance the impressed force, because at any intermediate points the value of $\frac{dy}{dx}$ is zero. Let the length of contact be from $x = a - b$ to $x = a + b$ and let $y_0^{\prime p \prime}$ be the value of y at any point within the *length of contact* $(2b)$.

[1] Kar—Phil. Mag., VI, *270*, (1928); Kar-Ghosh—Phil. Mag., IX, *806* (1930); Zeit für phys., 61, *525*. (1930); Phil. Mag., XII, *676* (1931); M. Ghosh—Ind. Phy. Math. Jour., III, *15*, (1932). For complete reference to the previous literatures, see Kar-Ghosh—Phil. Mag., IX, *806*.

[2] M. Ghosh—Phil. Mag., XVII, *521* (1934).

[3] Rayleigh—Theory of Sound, I, *191*.

The solutions of the differential equation of motion of the string with the conditions that (1) $y=0$ at $x=0$, $y=\gamma e^{ipt}$ at $x=a-b$, and (2) $y=\gamma e^{ipt}$ at $x=a+b$, $y=0$ at $x=l$, are

$$y_1 = \frac{\gamma}{\sin \lambda (a-b)} \sin \lambda x \, e^{ipt}, \tag{1}$$

from $x=0$ to $x=a-b$, and

$$y_2 = \frac{\gamma}{\sin \lambda (l-a-b)} \sin \lambda (l-x) \, e^{ipt}, \tag{2}$$

from $x=a+b$ to $x=l$.

If Fe^{ipt} be the periodic force imparted by the hammer, we have

$$F e^{ipt} = T_1 \left\{ \left(\frac{\partial y_1}{\partial x} \right)_{a-b} - \left(\frac{\partial y_2}{\partial x} \right)_{a+b} \right\}$$

$$= T_1 \gamma \lambda \left\{ \frac{\sin \lambda (l-2b)}{\sin \lambda (a-b) . \sin \lambda (l-a-b)} \right\} e^{ipt}, \tag{3}$$

where T_1 is the tension. Substituting the value of γ obtained from (3) in (1) and (2), and replacing the complex exponential by $\sin pt$, as the motion is started with initial velocity, we have,

$$y_1 = P \sin \lambda (l-a-b) \sin \lambda x \sin pt, \tag{4}$$

$$y_2 = P \sin \lambda (a-b) \sin \lambda (l-x) \sin pt, \tag{5}$$

where the arbitrary constant

$$P = \frac{F}{T_1 \lambda \sin \lambda (l-2b)}.$$

Since the force Fe^{ipt} is due to the reaction against the acceleration of the mass M, we have $F = M\gamma p^2$; so that from (3), we get

$$\rho \sin \lambda (l-2b) = M\lambda \sin\lambda (a-b) \sin\lambda (l-a-b), \tag{6}$$

where M is the mass of the load, ρ is the linear density of the string and $p=c\lambda$, c being the velocity of the wave. If the hammer strikes very near one end, *i.e.*, $a \rightarrow o$ and as λ is small we write $\lambda(a-b)$ for

$\sin\lambda(a-b)$ and l for $(l-2b)$ and also for $(l-a-b)$; then from (6), we get

$$\lambda = \sqrt{\frac{\rho}{M\,(a-b)}} = \frac{1}{c}\sqrt{\frac{T_1}{M\,(a-b)}} \tag{7}$$

The duration of contact Φ is given by

$$\Phi = \pi\sqrt{\frac{M\,(a-b)}{T_1}}. \tag{8}$$

Next we proceed to evaluate the arbitrary constant P. On differentiating (4) and (5) with respect to time t, we have at $t=0$,

$$(\dot{y}_1)_0 = Pc\lambda \sin\lambda(l-a-b)\sin\lambda x, \tag{9}$$
$$(\dot{y}_2)_0 = Pc\lambda \sin\lambda(a-b)\sin\lambda(l-x). \tag{10}$$

As *it is not a Fourier's series*, the arbitrary constant can be easily evaluated after Kar [1], by multiplying with ρdx and integrating. Then from eqs. (9) and (10), we have respectively,

$$\int_0^a \rho(\dot{y}_1)_0\,dx = Pc\lambda \sin\lambda\,(l-a-b)\int_0^a \rho\sin\lambda x\,dx, \tag{11}$$
$$\int_a^l \rho(\dot{y}_2)_0\,dx = Pc\lambda \sin\lambda\,(a-b)\int_a^l \rho\sin\lambda\,(l-x)\,dx. \tag{12}$$

As $(\dot{y}_1)_0$ and $(\dot{y}_2)_0$ have values only between the range $x=a-b$, to $a+b$, we get in the usual way[1]

$$\rho_0\int_{a-b}^{a} (\dot{y}_1)_0\,dx = Pc\lambda\sin\lambda\,(l-a-b)\,\Big[\rho_0\int_{a-b}^{a}\sin\lambda x\,dx + \rho\int_0^{a-b}\sin\lambda x\,dx\Big] \tag{13}$$

$$\rho_0\int_{a}^{a+b} (\dot{y}_2)_0\,dx = Pc\lambda\sin\lambda(a-b)\,\Big[\rho_0\int_{a}^{a+b}\sin\lambda\,(l-x)\,dx + \rho\int_{a+b}^{l}\sin\lambda\,(l-x)\,dx\Big], \tag{14}$$

where ρ_0 is the linear density of the string within the length of contact and ρ its linear density between the ranges 0 to $a-b$ and $a+b$ to l. After integrating (13) and (14) and adding, we get,

$$MV = Pc\rho\Big[\sin\lambda(a-b) + \sin\lambda\,(l-a-b) - \sin\lambda\,(l-2b) + \frac{\rho_0}{\rho}\sin\lambda(l-2b) - \frac{\rho_0}{\rho}\sin\lambda\,(l-a-b)\cos\lambda a - \frac{\rho_0}{\rho}\sin\lambda(a-b)\cos\lambda\,(l-a)\Big] \tag{15}$$

1 Kar—Ind. Phy. Math. Jour., III, *108*, (1932).

where M is the mass of the hammer, V is the initial velocity of the same. Writing $\cos\lambda\,(\overline{a-b}+b)$ and $\cos\lambda\,(\overline{l-a-b}+b)$ for $\cos\lambda a$ and $\cos\lambda\,(l-a)$ respectively and expanding $\sin\lambda b$ and $\cos\lambda b$, neglecting the higher terms than b^2 and with the help of eq. (6) the above equation reduces to

$$MV = Pc\rho\left[\sin\lambda\,(a-b)+\sin\lambda\,(l-a-b) + \frac{\rho_0}{\rho}\cdot\frac{\lambda^2 b^2}{2}\sin\lambda\,(l-2b)\right]. \tag{16}$$

Substituting the value of λ^2 in (16) from eq. (7) and as $2b\rho_0=M$, we have after substituting the value of P from (16),

$$y_1 = \frac{MV}{c\rho}\cdot\frac{\sin\lambda\,(l-a-b)\sin\lambda x\cdot\sin c\lambda t}{\sin\lambda\,(a-b)+\sin\lambda\,(l-a-b)+\dfrac{b}{4(a-b)}\sin\lambda(l-2b)}, \tag{17}$$

$$y_2 = \frac{MV}{c\rho}\cdot\frac{\sin\lambda\,(a-b)\sin\lambda\,(l-x)\sin c\lambda t}{\sin\lambda\,(a-b)+\sin\lambda\,(l-a-b)+\dfrac{b\sin\lambda(l-2b)}{4(a-b)}}. \tag{18}$$

And the displacement of the load is given by

$$y_0 = \frac{MV}{c\rho}\cdot\frac{\sin\lambda\,(l-a-b)\sin\lambda\,(a-b)\sin c\lambda t}{\sin\lambda\,(a-b)+\sin\lambda\,(l-a-b)+\dfrac{b\sin\lambda(l-2b)}{4(a-b)}}. \tag{19}$$

The equation (19) is further simplified by writing l for $l-a-b$ and also for $(l-2b)$ and $\lambda\,(a-b)$ for $\sin\lambda\,(a-b)$ and λl for $\sin\lambda l$.

The approximate solution becomes

$$y_0 = \frac{V}{c\lambda}\cdot\frac{\sin c\lambda t}{\left\{1+\dfrac{b}{4(a-b)}\right\}}, \tag{20}$$

where λ involves also 'b' (eq. 7), but as substitution of the value of λ from eq (7), involves no simplification, the expression is put as such to avoid the square-root sign. It is easily seen that if 'b', *which can*

not have values greater than 'a', be infinitesimally small, so that terms involving 'b' can be neglected, eqs. (7) and (20), also (17) and (18) reduce to the formulae obtained by Kar-Ghosh.[1] We may also note that in their formulae 'a' is taken to be the length of the shorter segment, but we can interpret it as '$a-b$' of the present investigation, because actually the portion of the string from '$a-b$' to 'a' is not free to vibrate as a part of the shorter segment.

It is also worth noting that if $b=a$, *i.e.*, if the hammer's breadth extends up to the nearer bridge, then from (8) and (20), $\Phi=0$ and $y_0=0$. In this case the hammer strikes the wire and rebounds back immediately without effecting any displacement of the string. If however 'b' decreases from the above limiting value, both Φ and y_0 gradually increase till they attain the maximum values given by Kar-Ghosh for $b=0$. Thus *the finite breadth of the hammer has the important effect on the duration of contact as well as on the maximum displacement.*

It is evident that in the present investigation, the plane of curvature of the hammer head is taken perpendicular to the length of the string. If, however, they are in the same plane, which is actually the case in a piano, the value of 'b' varies continually from zero to a maximum and then decreases to zero when the contact ceases. The rigorous solution of the problem is in progress, and will be communicated in a later issue. But a rough method may be given here in which we assume that instead of 'b' varying continually from 0 to b_{max}, and then to 0, the hammer has an average length of contact $\frac{1}{2}\,b_{max}$ throughout the duration of contact. Now 'b' attains its maximum value when y_0 is maximum. So we can write from geometrical conditions,

$$\tfrac{1}{2}\,b_{max.} = \tfrac{1}{2}\,\frac{rl\,(y_0)_{max}}{a(l-a)}\,, \tag{21}$$

where r is the radius of curvature of the hammer. Substituting the value of $(y_0)_{max}$ from eq. (20) in (21), we get approximately (neglecting the correction in λ and $(y_0)_{max}$ for finite length of contact and also writing l for $l-a$),

$$\tfrac{1}{2}\,b_{max.} = \tfrac{1}{2}\,\frac{r}{a}\cdot\frac{V}{\sqrt{\dfrac{T_1}{Ma}}} = \tfrac{1}{2}\left(\frac{M}{aT_1}\right)^{\frac{1}{2}} V\,r. \tag{22}$$

[1] Kar-Ghosh—Phil. Mag., IX, 806 (1930).

Substituting this value for 'b' in eq. (8), we get

$$\Phi = \pi \left\{ \frac{M \left(a - \frac{1}{2} \sqrt{\frac{M}{aT_1}} \cdot Vr\right)}{T_1} \right\}^{\frac{1}{2}}. \tag{28}$$

We find from eq. (28), that Φ decreases as 'r' or 'V' increases.

The experimental side of the problem of the pianoforte has been studied by many workers[1] including Kar-Ghosh.[2] But the effect of the breadth of the hammer on the duration of contact has not yet been studied thoroughly. As regards the relation of Φ and 'r', Ghosh[3] has found that for the mid-point Φ increases as 'r' increases. But it will be enough to point out from the discontinuous manner in which Φ changes with 'a' (vide page 372 Ghosh's paper) that the present problem for 'near end' bears no relation with the problem studied by Ghosh. However, the decrease of y_0 with the increase of 'r', as predicted by the theory is supported qualitatively by the experiment of Kar and others.[4] As their experimental measurements are not very accurate, a quantitative agreement cannot be expected.

We may remark that the variation of Φ with 'V' given by (28) must not be confused with a similar variation observed by Kaufmann[5] and Ghosh[6] in the case of felt hammer. Ghosh[6] has later on studied the problem theoretically by applying Hertz's law of impact holding for a certain time during contact. But even in his theory the effect of the length of contact (which increases from 0 to some maximum value) is not taken into account. The author hopes to discuss the effect of the length of contact at the centre, in a different communication.

In conclusion, I must express my sincerest thanks to Prof. K. C. Kar, D.Sc. for suggesting to me the problem when I drew his attention to the range of applicability of his own theory, and for his continual guidance and inspiring suggestions in the course of this work and also I must extend the same to Prof. M. Ghosh, M.Sc., A. Inst. P., for the interest he took in the work.

1 For complete reference, *vide* Ghosh—Ind. Jour. Phy. **VII** (1932), *365*.

2 Kar-Ghosh—Zeit. f. Phys., **66** (1930), *414*.

3 M. Ghosh, *loc. cit.*

4 Kar, Ganguly, Laha, Phil. Mag. **V** (1928), *547*.

5 Kaufmann. Ann. d. Phy. **54** (1895), *675*.

6 M. Ghosh, Phil. Mag. **XVII** (1934), *521*.

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15

On the solution of Laplace's equation suitable for problems relating to two spheres touching each other.

By

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Introduction.

In a previous paper,* Jeffery has given the solution of Laplace's equation suitable for problems with prescribed conditions on the surfaces of two spheres, which neither intersect nor touch each other. He starts with two systems of orthogonal circles in the plane of xz, the members of the first system passing through two fixed points on the z axis, and those of the second system having these fixed points as their limiting points. His coordinates are then obtained by rotation about the axis of z, so that one of the systems of circles, generates a system of coaxal spheres with a common radical plane. But, if we start with two orthogonal systems of circles in the xz plane, touching the axes of x and z respectively, then by rotation about the axis of z, the system of circles touching the x-axis, will generate a series of spheres touching the plane $z = 0$. Employing these coordinates, we get in this paper, a solution of Laplace's equation, suitable for problems, where the boundaries are two spheres touching each other.

Coordinates.

Let a system of coordinates α, β, γ be defined by

$$\alpha + i\beta = \frac{a}{z+i\rho}, \quad \gamma = \tan^{-1}\frac{y}{x}, \tag{1}$$

where $\rho = \sqrt{(x^2+y^2)}$. Then

$$\alpha = \frac{az}{\rho^2+z^2}, \quad \beta = -\frac{a\rho}{\rho^2+z^2}, \tag{2}$$

$$z = \frac{a\alpha}{\alpha^2+\beta^2}, \quad \rho = -\frac{a\beta}{\alpha^2+\beta^2}. \tag{3}$$

* Proc. Roy. Soc. (Ser. A), 87, 109.

The surfaces γ = constant are planes passing through the z-axis. In any meridianal plane γ, the curves α = constant and β = constant are two systems of orthogonal circles, touching the axes of z and ρ respectively, at the origin. The surfaces α = constant are, therefore, spheres touching the plane of xy, at the origin.

We have

$$\frac{1}{h^2} = \left(\frac{\partial\rho}{\partial\alpha}\right)^2 + \left(\frac{\partial\rho}{\partial\beta}\right)^2 = \frac{a^2}{(a^2+\beta^2)^2},$$

so that

$$h = \frac{a^2+\beta^2}{a} = -\frac{\beta}{\rho}. \tag{4}$$

Solution of Laplace's equation.

In orthogonal curvilinear coordinates, α, β, γ, we have

$$\nabla^2\phi = h_1 h_2 h_3 \left[\frac{\partial}{\partial\alpha}\left(\frac{h_1}{h_2 h_3}\frac{\partial\phi}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_2}{h_3 h_1}\frac{\partial\phi}{\partial\beta}\right) + \frac{\partial}{\partial\gamma}\left(\frac{h_3}{h_1 h_2}\frac{\partial\phi}{\partial\gamma}\right)\right]$$

where

$$h_1^2 = \left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2,$$

$$h_2^2 = \left(\frac{\partial\beta}{\partial x}\right)^2 + \left(\frac{\partial\beta}{\partial y}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2,$$

$$h_3^2 = \left(\frac{\partial\gamma}{\partial x}\right)^2 + \left(\frac{\partial\gamma}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2,$$

Here

$$h_1^2 = \left(\frac{\partial\alpha}{\partial\rho}\right)^2 + \left(\frac{\partial\alpha}{\partial z}\right)^2 = h^2,$$

$$h_2^2 = \left(\frac{\partial\beta}{\partial\rho}\right)^2 + \left(\frac{\partial\beta}{\partial z}\right)^2 = h^2,$$

$$h_3^2 = \frac{1}{\rho^2},$$

so that Laplace's equation

$$\nabla^2\phi = 0, \tag{5}$$

reduces to

$$\frac{\partial}{\partial \alpha}\left(\rho\, \frac{\partial \phi}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\rho\frac{\partial \phi}{\partial \beta}\right)+\frac{1}{\rho h^2}\, \frac{\partial^2 \phi}{\partial \gamma^2} = 0. \tag{6}$$

Substituting $\phi = \chi/\sqrt{\rho}$ in (6), and simplifying, we find

$$\frac{\partial^2\chi}{\partial \alpha^2}+\frac{\partial^2\chi}{\partial \beta^2}-\frac{\chi}{2\rho}\left(\frac{\partial^2\rho}{\partial \alpha^2}+\frac{\partial^2\rho}{\partial \beta^2}\right)+\frac{\chi}{4\rho^2}\left\{\left(\frac{\partial \rho}{\partial \alpha}\right)^2+\left(\frac{\partial \rho}{\partial \beta}\right)^2\right\}$$

$$+\frac{1}{\rho^2 h^2}\,\frac{\partial^2\chi}{\partial \gamma^2} = 0. \tag{7}$$

Remembering that $\rho + iz$ is a function of $\alpha + i\beta$, we have

$$\frac{\partial^2\rho}{\partial \alpha^2}+\frac{\partial^2\rho}{\partial \beta^2} = 0\,, \quad \left(\frac{\partial \rho}{\partial \alpha}\right)^2+\left(\frac{\partial \rho}{\partial \beta}\right)^2 = \frac{1}{h^2}\,,$$

so that (7) reduces to

$$\frac{\partial^2\chi}{\partial \alpha^2}+\frac{\partial^2\chi}{\partial \beta^2}+\frac{1}{\rho^2h^2}\left(\frac{\partial^2\chi}{\partial \gamma^2}+\frac{\chi}{4}\right) = 0. \tag{8}$$

Let us seek a solution of the equation (8), of the type

$$\chi = \mathrm{UV}\cos(m\gamma + c),$$

where U is a function of α alone, V a function of β alone and m is a positive integer. Substituting this value of χ in (8), and simplifying, we get

$$\frac{1}{\mathrm{U}}\,\frac{d^2\mathrm{U}}{d\alpha^2}+\frac{1}{\mathrm{V}}\,\frac{d^2\mathrm{V}}{d\beta^2}+\frac{1}{\rho^2h^2}\left(\frac{1}{4}-m^2\right) = 0.$$

If $1/\rho^2h^2$ is of the form $f(\alpha)+\mathrm{F}(\beta)$, then

$$-\frac{1}{\mathrm{U}}\,\frac{d^2\mathrm{U}}{d\alpha^2}-f(\alpha)\left(\frac{1}{4}-m^2\right) = \frac{1}{\mathrm{V}}\,\frac{d^2\mathrm{V}}{d\beta^2}+\mathrm{F}(\beta)\left(\frac{1}{4}-m^2\right)$$

$$= \text{constant},$$

so that U and V are known.

In our case, $\rho h = -\beta$, so that U and V are determined from the equations,

$$-\frac{1}{U}\frac{d^2U}{d\alpha^2} = \frac{1}{V}\frac{d^2V}{d\beta^2} + \frac{1-4m^2}{4\beta^2} = \text{constant} = -n^2 \text{ (say)}.$$

Then $\qquad U = A_n \cosh n\alpha + B_n \sinh n\alpha,$

and

$$\frac{d^2V}{d\beta^2} + \left(n^2 + \frac{1-4m^2}{4\beta^2}\right)V = 0.$$

Putting $\quad V = \sqrt{\beta}.\, v$, this equation becomes

$$\frac{d^2v}{d\beta^2} + \frac{1}{\beta}\frac{dv}{d\beta} + \left(n^2 - \frac{m^2}{\beta^2}\right)v = 0,$$

so that

$$v = a_n' J_m(n\beta) + b_n' Y_m(n\beta),$$

where J_m and Y_m are Bessel's functions of order m, of the first and the second kinds respectively.

Therefore, we have

$$\phi = \sqrt{(\alpha^2+\beta^2)} \sum_{m=0}^{\infty} \cos(m\gamma + c) \sum_{n=0}^{\infty} [\{A_{m,n} \cosh n\alpha + B_{m,n} \sinh n\alpha\}$$

$$\times J_m(n\beta) + \{C_{m,n} \cosh n\alpha + D_{m,n} \sinh n\alpha\} Y_m(n\beta)]. \quad (9)$$

Stokes' Stream function.

In the case of symmetry, about the z-axis,

$$\phi = \sqrt{(\alpha^2+\beta^2)} \sum_{n=0}^{\infty} [\{A_n \cosh n\alpha + B_n \sinh n\alpha\} J_0(n\beta)$$

$$+ \{C_n \cosh n\alpha + D_n \sinh n\alpha\} Y_0(n\beta)]. \quad (10)$$

If ψ be Stokes' stream function,

$$\frac{\partial\phi}{\partial z} = \frac{1}{\rho}\frac{\partial\psi}{\partial\rho}, \quad \frac{\partial\phi}{\partial\rho} = -\frac{1}{\rho}\frac{\partial\psi}{\partial z},$$

so that

$$\frac{\partial \phi}{\partial \alpha} = \frac{\partial \phi}{\partial \rho} \cdot \frac{\partial \rho}{\partial \alpha} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial \alpha}$$

$$= \frac{1}{\rho} \frac{\partial \psi}{\partial z} \cdot \frac{\partial z}{\partial \beta} + \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \cdot \frac{\partial \rho}{\partial \beta}$$

$$= \frac{1}{\rho} \frac{\partial \psi}{\partial \beta},$$

$$\frac{\partial \phi}{\partial \beta} = \frac{\partial \phi}{\partial \rho} \cdot \frac{\partial \rho}{\partial \beta} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial z}{\partial \beta}$$

$$= -\frac{1}{\rho} \frac{\partial \psi}{\partial z} \cdot \frac{\partial z}{\partial \alpha} - \frac{1}{\rho} \frac{\partial \psi}{\partial \rho} \cdot \frac{\partial \rho}{\partial \alpha}$$

$$= -\frac{1}{\rho} \frac{\partial \psi}{\partial \alpha}.$$

Assuming ϕ to be continuous, we have on elimination of ϕ,

$$\frac{\partial}{\partial \alpha}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \alpha}\right) + \frac{\partial}{\partial \beta}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial \beta}\right) = 0. \tag{11}$$

Putting $\psi = \sqrt{\rho} . \psi'$, this equation becomes

$$\frac{\partial^2 \psi'}{\partial \alpha^2} + \frac{\partial^2 \psi'}{\partial \beta^2} - \frac{3\psi'}{4\beta^2} = 0.$$

If we assume $\psi' = UV$, where U is a function of α alone and V a function of β alone, we get

$$-\frac{1}{U} \frac{d^2 U}{d\alpha^2} = \frac{1}{V} \frac{d^2 V}{d\beta^2} - \frac{3}{4\beta^2}$$

$$= \text{constant} = -n^2 \text{ (say)},$$

so that

$$U = a_n \cosh n\alpha + b_n \sinh n\alpha,$$

and

$$\frac{d^2V}{d\beta^2}+\left(n^2-\frac{3}{4\beta^2}\right)V = 0. \tag{12}$$

Substituting $V = \sqrt{\beta}.v$, (12) becomes

$$\frac{d^2v}{d\beta^2}+\frac{1}{\beta}\frac{dv}{d\beta}+\left(n^2-\frac{1}{\beta^2}\right)v = 0,$$

so that
$$v = a'_n J_1(n\beta)+b'_n Y_1(n\beta).$$

Therefore

$$\psi = \frac{\beta}{\sqrt{(\alpha^2+\beta^2)}}\sum_{n=0}^{\infty}\Big[(a_n \cosh n\alpha+b_n \sinh n\alpha)\, J_1\,(n\beta)$$

$$+(c_n \cosh n\alpha+d_n \sinh n\alpha)\, Y_1\,(n\beta)\Big]. \tag{13}$$

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NOTE ON THE TRANSVERSE VIBRATION OF FREELY SUPPORTED RECTANGULAR PLATES UNDER THE ACTION OF MOVING LOADS AND VARIABLE FORCES

BY

BIBHUTIBHUSAN SEN.

1. *Introduction.*

The transverse oscillation of beams under the action of moving loads has been the subject of considerable research * in connection with the study of bridge vibrations. The object of this note is to consider similar problems in the case of a thin rectangular plate, the edges being freely supported. The vibration problems of such plates without any load have simple solutions and have been thoroughly treated by Lord Rayleigh. †

In this paper we take

$$\begin{array}{ll} \text{the thickness of the plate} & = 2h, \\ \text{the Young's modulus} & = E, \\ \text{the Poisson's ratio} & = \sigma, \\ \text{the density of the material} & = \rho, \\ \text{the small normal deflection} & = w, \text{ and} \\ \text{the transverse force on an} & \\ \qquad\qquad \text{element of area} & = Z.\ \rho.\ 2h, \end{array} \tag{1·1}$$

* See, for example, the paper "On the forced vibration of bridges," by S. P. Timoshenko in the Philosophical Magazine, Series 6, 44, (1922), *1018* and also the paper "On transverse oscillations of beams under the action of moving loads" by A. N. Lowan in the Philosophical Magazine, series 7, 19 (1935), *708*.

† Theory of Sound, 1, *371*.

Then the equation satisfied by the transverse displacement w is

$$\frac{\mathrm{E}\,h^2}{3\rho\,(1-\sigma^2)}\,\nabla_1{}^4 w - \mathrm{Z} + \frac{d^2 w}{dt^2} = 0,$$

or

$$\mathrm{D}\,\nabla_1{}^4 w - \phi\,(x,\,y,\,t) + \frac{d^2 w}{dt^2} = 0, \tag{1·2}$$

where

$$\mathrm{D} \text{ stands for } \frac{\mathrm{E}\,h^2}{3\rho\,(1-\sigma^2)},$$

and

$$\phi\,(x,\,y,\,t) \text{ stands for Z}. \tag{1·3}$$

If the boundaries of the plate be taken as the lines

$$x = 0,\ x = a,\ y = 0 \text{ and } y = b,$$

conditions for supported edge are

$$\left.\begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial x^2} + \sigma\,\frac{\partial^2 w}{\partial y^2} = 0 \end{aligned}\right\} \begin{aligned} &\text{when } x = o \\ &\text{and } \quad x = a, \end{aligned} \tag{1·4}$$

$$\left.\begin{aligned} & w = 0 \\ & \frac{\partial^2 w}{\partial y^2} + \sigma\,\frac{\partial^2 w}{\partial x^2} = 0 \end{aligned}\right\} \begin{aligned} &\text{when } y = o \\ &\text{and } \quad y = b. \end{aligned} \tag{1·5}$$

2. *General solution of the equation.*

If we assume

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathrm{Q}_{mn}\,(t)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}, \tag{2·1}$$

where $\mathrm{Q}_{mn}\,(t)$ is a function of t and 'm' and 'n' are positive integers, we find that the boundary conditions given in (1·4) and (1·5) are satisfied.

We now suppose that it is possible to express $\phi\,(x,\,y,\,t)$ in the form

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \psi_{mn}\,(t)\,\sin\frac{m\pi x}{a}\,\sin\frac{n\pi y}{b}. \tag{2·2}$$

Then the equation (1·2) will be satisfied if

$$D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2 Q_{mn}(t)-\psi_{mn}(t)+\frac{d^2 Q_{mn}(t)}{dt^2}=0,$$

i.e., if

$$\frac{d^2 Q_{mn}(t)}{dt^2}+p_{mn}^2 Q_{mn}(t)=\psi_{mn}(t), \tag{2·3}$$

in which

$$p_{mn}^2=D\left(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}\right)^2. \tag{2·4}$$

The general solution of the equation (2·3) is

$$Q_{mn}(t)=\frac{1}{p_{mn}}\int_0^t \sin p_{mn}(t-u)\,\psi_{mn}(u)\,du$$

$$+A_{mn}\cos p_{mn}t+B_{mn}\sin p_{mn}t, \tag{2·5}$$

A_{mn} and B_{mn} being arbitrary constants.

3. *Solution for a pulsating force acting at the point* (ξ,η).

If $\phi(x,y,t)=Z_0\sin pt$ at the point (ξ,η) and zero elsewhere, we have by harmonic analysis

$$\psi_{mn}(t)=\frac{4Z_0}{ab}\sin pt\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}, \tag{3·1}$$

which makes

$$Q_{mn}(t)=\frac{4Z_0}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\frac{1}{p_{mn}}\int_0^t \sin p_{mn}(t-u)\sin pu\,du$$

$$=\frac{4Z_0}{ab}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}\frac{[p_{mn}\sin pt-p\sin p_{mn}t]}{p_{mn}(p_{mn}^2-p^2)}. \tag{3·2}$$

[The arbitrary constants A_{mn} and B_{mn} will be zero, if it is assumed that $w=\frac{dw}{dt}=0$, when $t=0$.]

Then the expression for w will be

$$w=\frac{4Z_0}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{p_{mn}\sin pt-p\sin p_{mn}t}{p_{mn}(p_{mn}^2-p^2)}\sin\frac{m\pi\xi}{a}\sin\frac{n\pi\eta}{b}$$

$$\times\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{3·3}$$

If for certain values of m and n

$$p_{mn} = p\,,$$

the expression (8·2) is indeterminate. By writing

$$p = p_{mn} - p',$$

where p' is very small, we obtain from (8·2),

$$Q_{mn}(t) = \frac{4Z_o}{ab} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b}$$

$$\times \frac{[p_{mn}(\sin p_{mn}t - p't\cos p_{mn}t) - (p_{mn}-p')\sin p_{mn}t]}{p_{mn}\,p'\,(2p_{mn}-p')}$$

to the first order in p'.

Making p' tend to zero, we have

$$Q_{mn}(t) = \frac{2Z_o}{ab\,p_{mn}^2} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \cdot [\sin p_{mn}t - p_{mn}t\cos p_{mn}t], \tag{8·4}$$

in *the special case* when a pulsating load resonates with a natural frequency of the plate.

The term $p_{mn}(p_{mn}^2 - p^2)$ in the denominator of (8·3) shows that if the applied frequency is less than the gravest natural frequency of the plate, the series is convergent, and the first term in the series, namely, that one obtained by putting $m=1$ and $n=1$, will be predominant in the complete expression for the deflection. Thus, unless there be resonance, we can write as a first approximation

$$w = \frac{4Z_o}{ab\,p_{11}(p_{11}^2 - p^2)}[p_{11}\sin pt - p\sin p_{11}t]\ \sin\frac{\pi\xi}{a}\sin\frac{\pi\eta}{b}$$

$$\times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}, \tag{8·5}$$

where

$$p_{11} = \sqrt{D}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\pi^2. \tag{8·6}$$

If the frequency of the applied force resonates with the gravest natural frequency of the plate, we have approximately

$$w = \frac{2Z_o}{ab\,p_{11}^2}\ [\sin p_{11}t - p_{11}t\cos p_{11}t]\ \sin\frac{\pi\xi}{a}\sin\frac{\pi\eta}{b}$$

$$\times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{8·7}$$

We find in (8·2) that there are two terms within the bracket, of which one is proportional to $\sin pt$ and the other proportional to $\sin p_{mn} t$. The former gives the same period as the disturbing force and represents forced vibrations of the plate. The latter represents free vibrations of the plate produced by the application of the force. These latter vibrations due to various kinds of resistance will be gradually damped out and only the forced vibrations will persist. Then we shall have

$$w = \frac{4Z_0}{ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{p_{mn}{}^2 - p^2} \sin\frac{m\pi\xi}{a} \sin\frac{n\pi\eta}{b} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \times \sin pt. \tag{8·8}$$

If the pulsating force Z is varying very slowly, p is a very small quantity. Then neglecting p^2 in comparison with $p_{mn}{}^2$ and putting $m=1$, $n=1$, we have for the constant load Z, the approximate value* of

$$w = \frac{4Z}{ab\, p_{11}{}^2} \sin\frac{\pi\xi}{a} \sin\frac{\pi\eta}{b} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{8·9}$$

In the particular case when the force is applied at the middle of the plate, the deflection at the centre is approximately

$$= \frac{4Z}{ab\, p_{11}{}^2}. \tag{8·10}$$

Again, retaining only the first term of (8·8), we have

$$w = \frac{4Z}{ab\,(p_{11}{}^2 - p^2)} \sin\frac{\pi\xi}{a} \sin\frac{\pi\eta}{b} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{8·11}$$

Hence we conclude from this result and (8·9) that the ratio of $(w)_d$, the dynamical deflection to $(w)_s$, the statical deflection is roughly

$$\frac{(w)_d}{(w)_s} = \frac{1}{1 - p^2/p_{11}^2}. \tag{8·12}$$

Let us suppose, for instance, that the frequency of the fundamental mode of vibration is five times as great as the frequency of the disturbing force. Then the dynamical deflection will be about 4 per cent. greater than the statical deflection. Proceeding in a similar way, we can find the effect of several pulsating forces acting on the plate, as the resulting vibration will be obtained by superposing the vibration produced by the individual forces.

* This result for the constant load can be deduced directly.

4. *Continuously distributed pulsating force.*

If the plate be loaded by a uniformly distributed load of intensity

$$Z = Z_0 \sin pt\ , \tag{4·1}$$

where Z_0 is constant, we can deduce the expression for the deflection from the expression obtained for a pulsating force applied at a particular point. Multiplying the right side of (3·2) by $d\xi\, d\eta$ and integrating with respect to ξ and η between the limits o to a and o to b respectively, we get

$$Q_{mn}(t) = \frac{4Z_0}{mn\pi^2} \frac{(1-\cos m\pi)(1-\cos n\pi)}{p_{mn}(p_{mn}^2 - p^2)} [p_{mn} \sin pt - p \sin p_{mn} t],$$

which gives

$$w = \frac{4Z_0}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(1-\cos m\pi)\ (1-\cos n\pi)}{mnp_{mn}\ (p_{mn}^2 - p^2)}$$

$$\times [p_{mn} \sin pt - p \sin p_{mn} t] \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}. \tag{4·2}$$

This series converges very rapidly and on retaining only the first term, we arrive at the same conclusion as in (3·12).

5. *Oscillation due to a constant moving load.*

Let us now consider the problem of a constant load moving parallel to the axis of x across the rectangular plate with constant speed v, the load starting from the point $(0,\ b/2)$. Then we can write

$$\psi_{mn}(t) = \frac{4Z_0}{ab} \sin \frac{m\pi\xi}{a} \sin \frac{n\pi\eta}{b}\ , \tag{5·1}$$

where

$$Z_0 \times \rho \times 2h = \text{load per unit area,}$$

$$\xi = vt, \quad \text{and}$$

$$\eta = \frac{b}{2}.$$

Assuming that $w = \dfrac{dw}{dt} = 0$ when $t=0$, we have from (2·5),

$$Q_{mn}(t) = \frac{4Z_0 \sin n\pi/2}{p_{mn}ab} \int_0^t \sin\ p_{mn}(t-u)\ \sin \frac{m\pi v}{a} u\, du$$

$$= \frac{4Z_0\ \sin\ n\pi/2}{p_{mn}ab\left(p_{mn}^2 - \frac{m^2\pi^2v^2}{a^2}\right)}\left[p_{mn}\sin\ \frac{m\pi vt}{a} - \frac{m\pi v}{a}\sin p_{mn}t\right], \tag{5·2}$$

whence we obtain

$$w = \frac{4Z_0}{ab}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi}{2}}{p_{mn}\left(p_{mn}^2 - \frac{m^2\pi^2v^2}{a^2}\right)}$$

$$\times\left[p_{mn}\sin\frac{m\pi vt}{a} - \frac{m\pi v}{a}\sin p_{mn}t\right]\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}. \tag{5·3}$$

If the velocity v be small, we have approximately

$$w = \frac{4Z_0}{ab}\ \frac{1}{p_{11}\left(p_{11}^2 - \frac{\pi^2v^2}{a^2}\right)}\left[p_{11}\ \sin\ \frac{\pi vt}{a} - \frac{\pi v}{a}\sin\ p_{11}t\right]$$

$$\times \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{5·4}$$

Assuming that in the most unfavourable case the contributions made by the amplitudes of $\sin\frac{\pi vt}{a}$ and $\sin p_{11}t$ are added to one another, we obtain for the maximum deflection

$$(w)_{max} = \frac{4Z_0}{ab\ p_{11}\left(p_{11}^2 - \frac{\pi^2v^2}{a^2}\right)}\left(p_{11} + \frac{\pi v}{a}\right)$$

$$= \frac{4Z_0}{ab\left(p_{11} - \frac{\pi v}{a}\right)p_{11}}. \tag{5·5}$$

This result has been obtained on the hypothesis that there is no damping and hence it is evident that this value of maximum deflection is somewhat exaggerated.

By increasing the velocity v, a condition can be reached when one of the denominators in the series (5·3) becomes equal to zero and resonance takes place.

Suppose $$p_{11} = \frac{\pi v}{a}. \tag{5·6}$$

Proceeding as in section 3, we obtain from (5·4)

$$w = \frac{2Z_0}{abp_{11}^2}\left[\sin p_{11}t - p_{11}t\cos p_{11}t\right]\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{5·7}$$

This expression has a maximum value when

$$t = \frac{\pi}{p_{11}} = \frac{v}{a}. \tag{5·8}$$

Then

$$w = \frac{2Z_0\pi}{abp_{11}^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}. \tag{5·9}$$

We find from (5·8) that this maximum deflection takes place when the load is leaving the plate and at that instant its value at the centre is

$$(w)_r = \frac{2Z_0\pi}{abp_{11}^2} = \frac{2Z_0\pi}{v^2}\,\frac{a}{b}. \tag{5·9}$$

On comparison with (3·10) we find

$$\frac{(w)_r}{(w)_s} = \frac{\pi}{2}. \tag{5·10}$$

We can also obtain the deflection when the load moves along any other line across the rectangular plate. As for instance, putting $\xi = v_1 t$ and $\eta = v_2 t$ in (5·1), we can determine the displacement when the load moves from the origin with the velocity which has components v_1 and v_2 parallel to the x-axis and y-axis respectively.

In this case

$$Q_{mn}(t) = \frac{4Z_0}{abp_{mn}}\int_0^t \sin p_{mn}(t-u)\sin\frac{m\pi v_1}{a}u\sin\frac{n\pi v_2}{b}u\,du$$

$$= \frac{2Z_0}{ab}\left[\left\{\frac{\cos\left(\frac{m\pi v_1}{a}-\frac{n\pi v_2}{b}\right)t}{p_{mn}^2-\left(\frac{m\pi v_1}{a}-\frac{n\pi v_2}{b}\right)^2}-\frac{\cos\left(\frac{m\pi v_1}{a}+\frac{n\pi v_2}{b}\right)t}{p_{mn}^2-\left(\frac{m\pi v_1}{a}+\frac{n\pi v_2}{b}\right)^2}\right\}\right.$$

$$\left.+\left\{\frac{1}{p_{mn}^2-\left(\frac{m\pi v_1}{a}+\frac{n\pi v_2}{b}\right)^2}-\frac{1}{p_{mn}^2-\left(\frac{m\pi v_1}{a}-\frac{n\pi v_2}{b}\right)^2}\right\}\cos p_{mn}t\right]. \tag{5·11}$$

$Q_{mn}(t)$ being found, the expression for w can be written from (2·1).

6. *Moving pulsating force.*

The next case that we shall consider is that of a pulsating force which travels along the plate parallel to the axis of x, starting from the point $(0, \frac{b}{2})$. Then we can write from (5·1)

$$\psi_{mn}(t) = \frac{4Z_0}{ab}\sin\Omega t\sin\frac{m\pi vt}{a}\sin\frac{n\pi}{2}, \tag{6.1}$$

where

$\frac{2\pi}{\Omega}$ = period of the pulsating force, and

v = the velocity of the load.

Substituting this value of $\psi_{mn}(t)$ in the first term of (2·5), we obtain

$$Q_{mn}(t) = \frac{4Z_0}{abp_{mn}}\sin\frac{n\pi}{2}\int_0^t \sin p_{mn}(t-u)\sin\Omega u\sin\frac{m\pi vu}{a}\,du$$

$$= \frac{2Z_0}{ab}\sin\frac{n\pi}{2}\left[\left\{\frac{\cos\left(\frac{m\pi v}{a}-\Omega\right)t}{p_{mn}^2-\left(\frac{m\pi v}{a}-\Omega\right)^2}-\frac{\cos\left(\frac{m\pi v}{a}+\Omega\right)t}{p_{mn}^2-\left(\frac{m\pi v}{a}+\Omega\right)^2}\right\}\right.$$

$$\left.+\left\{\frac{1}{p_{mn}^2-\left(\frac{m\pi v}{a}+\Omega\right)^2}-\frac{1}{p_{mn}^2-\left(\frac{m\pi v}{a}-\Omega\right)^2}\right\}\cos p_{mn}t\right]. \tag{6·2}$$

If the fluctuating pressure be due to the impact of the wheel running on the plate, Ω, the angular velocity of the wheel will be proportional to v. There are two harmonic components $\cos\left(\frac{m\pi v}{a} - \Omega\right)t$ and $\cos\left(\frac{m\pi v}{a} + \Omega\right)t$ in the above expression for $Q_{mn}(t)$. The frequency of the second component will resonate at a lower speed of travel than that of the first and hence the resonance will occur first when

$$\Omega + \frac{\pi v}{a} = p_{11}. \tag{6·3}$$

The deflection is then approximately given by

$$w = -\frac{Z_0}{abp_{11}} t \sin p_{11}t \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \tag{6·4}$$

it being assumed that $w = \frac{dw}{dt} = 0$, when $t = 0$.

The time taken by the load to cross the plate is a/v.

So, at the instant when the load is leaving the plate, we have

$$p_{11}t = \frac{\Omega a}{v} + \pi$$

and the deflection

$$w = \frac{Z_0}{abp_{11}^2}\left(\frac{\Omega a}{v} + \pi\right)\sin\frac{\Omega a}{v} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{6·5}$$

Hence, at the instant when the load is leaving the plate, the deflection at the centre is approximately

$$\frac{Z_0}{ab\, p_{11}^2}\left(\frac{\Omega a}{v} + \pi\right)\sin\frac{\Omega a}{v}. \tag{6·6}$$

7. *Conclusion.*

In the foregoing sections, several cases of forced vibrations of freely supported rectangular plates are considered for determining the deflection caused by the moving loads and variable forces. The approximate results given in (3·12), (5·5), (5·10) and (6·6) are of some practical interest and they have the same degree of accuracy as those obtained for the beams by the previous writers.

An infinite integral involving Bessel function and parabolic cylinder function

By

R. S. Varma.

The object of this note is to evaluate an infinite integral involving Bessel function and the parabolic cylinder function. We shall require the following Lemma:

$$\int_0^\infty x^n e^{-px+\frac{1}{4}x^2} D_{-m}(x)\, dx$$

$$= \frac{p^{m-n-1}}{\Gamma(m)} \sum_{r=0}^{\infty} \frac{(-\frac{1}{2}p^2)^r}{r\,!} \Gamma(m+2r)\Gamma(n-2r-m+1),$$

when $R(m)>0$, $R(n)>-1$ and $m-n$ is not an integer.

To prove this, we use Whittaker's integral* for $D_n(z)$, *viz.*,

$$D_n(z) = \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-tz-\frac{1}{2}t^2} t^{-n-1}\, dt, \qquad [R(n)<0].$$

We then have

$$\int_0^\infty x^n e^{-px+\frac{1}{4}x^2} D_{-m}(x) dx$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty x^n e^{-px} dx \int_0^\infty e^{-xt-\frac{1}{2}t^2} t^{m-1} dt, \quad [R(m)>0].$$

$$= \frac{1}{\Gamma(m)} \int_0^\infty t^{m-1} e^{-\frac{1}{2}t^2} dt \int_0^\infty e^{-x(t+p)} x^n\, dx,$$

* E. T. Whittaker: On the functions associated with the parabolic cylinder in harmonic analysis, Proc. Lond. Math. Soc. (1), 35 (1903), 417-427.

change in the order of integration being obviously permissible,

$$= \frac{\Gamma(n+1)}{\Gamma(m)} \int_0^\infty \frac{t^{m-1} e^{-\frac{1}{2}t^2}}{(p+t)^{n+1}}\, dt.$$

Expanding $e^{-\frac{1}{2}t^2}$ in powers of t and by the help of the integral

$$\int_0^\infty \frac{t^{l-1}}{(a+t)^{l+m}}\, dt = a^{-m} \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)},$$

integrating term by term, a process easily justifiable, we at once obtain the required lemma.

Now consider the integral

$$I = \int_0^\infty x^n e^{-px+\frac{1}{4}x^2} D_{-m}(x) J_\nu(xa)\, dx.$$

Expanding $J_\nu(xa)$, we get

$$I = \int_0^\infty x^n e^{-px+\frac{1}{4}x^2} D_{-m}(x) \sum_{s=0}^\infty \frac{(-)^s (ax)^{\nu+2s}}{2^{\nu+2s} s!\, \Gamma(\nu+s+1)}\, dx$$

$$= \sum_{s=0}^\infty \frac{(-)^s a^{\nu+2s}}{2^{\nu+2s}\, s!\, \Gamma(\nu+s+1)} \int_0^\infty x^{\nu+n+2s} e^{-px+\frac{1}{4}x^2} D_{-m}(x)\, dx,$$

the process involved being easily justifiable,

$$= \sum_{s=0}^\infty \frac{(-)^s a^{\nu+2s}}{2^{\nu+2s}\, s!\, \Gamma(\nu+s+1)} \frac{p^{m-n-\nu-2s-1}}{\Gamma(m)}$$

$$\times \sum_{r=0}^\infty \frac{(-\frac{1}{2}p^2)^r}{r!} \Gamma(m+2r)\, \Gamma(n+\nu+2s-2r-m+1),$$

where we have used our lemma established above.

The second series has a meaning so long as $m-n-\nu$ is not an integer. Since both the infinite series are absolutely convergent

when $|a| \leqq |p|$ and $|p| < 1$ we can write

$$I = \left(\frac{a}{2}\right)^{\nu} \frac{p^{m-n-\nu-1}}{\Gamma(m)} \sum_{r=0}^{\infty} \frac{(-\frac{1}{2}p^2)^r}{r!} \Gamma(2r+m)$$

$$\times \sum_{s=0}^{\infty} \frac{(-)^{\nu} a^{2s} p^{-2s} \Gamma(n+\nu+2s-2r-m+1)}{2^{2s}\, s!\, \Gamma(\nu+s+1)}$$

$$= \left(\frac{a}{2}\right)^{\nu} \frac{p^{m-n-\nu-1}}{\Gamma(m)\Gamma(\nu+1)} \sum_{r=0}^{\infty} \frac{(-\frac{1}{2}p^2)^r}{r!} \Gamma(n+\nu-2-m+1$$

$$\times\ {}_2F_1\left\{\begin{matrix} \frac{1}{2}n+\frac{1}{2}\nu-r-\frac{1}{2}m+\frac{1}{2},\ \frac{1}{2}n+\frac{1}{2}\nu-r-\frac{1}{2}m+1; \\ \nu+1 \end{matrix} -\frac{a^2}{p^2}\right\}$$

true when $R(m)>0$, $R(n)>-1$ and $m-n-\nu$ is not an integer.

In particular, when $a=ip$ and $m-n-\nu$ is of the type $2l+\frac{1}{2}$, l being an integer,

$$\int_0^{\infty} x^n e^{-px+\frac{1}{4}x^2}\, D_{-m}(x)\, J_{\nu}(ipx)dx$$

$$= \frac{\pi\, i^{\nu}\, (2p)^{m-n-1}\, \Gamma(m-n-\frac{1}{2})}{\Gamma(m)\Gamma(m-n-\nu)\Gamma(\nu-n+m)} \times$$

$${}_2F_4\left\{\begin{matrix} \frac{1}{2}m-\frac{1}{2}n-\frac{1}{4},\ \frac{1}{2}m-\frac{1}{2}n+\frac{1}{4} \\ \frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}\nu,\ \frac{1}{2}m-\frac{1}{2}n-\frac{1}{2}\nu+\frac{1}{2},\ \frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2}m,\ \frac{1}{2}\nu-\frac{1}{2}n+\frac{1}{2}m+\frac{1}{2} \end{matrix} ; -\frac{1}{2}p^2\right\}$$

On some two-dimensional problems of Elasticity

By

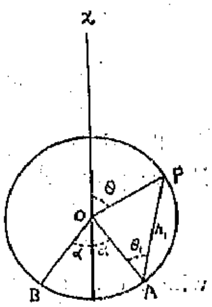
S. Ghosh.

Introduction.

Particular problems of generalised plane stress in annular circular plates under the action of isolated forces on the boundaries, have been considered by Filon* and Sen** in two important memoirs, mainly in connection with the determination of stresses in circular rings under the action of self-equilibriating forces on the rims. In the present paper, other interesting cases of generalised plane stress in circular discs, are considered, in which the weight of the disc is not neglected. In the first case that is considered here, the stress function is found, for a heavy circular disc, resting in a vertical plane on two smooth pegs, either at the same horizontal level or at two different horizontal levels. In the second case, a determination is made, of the stress function suitable for a heavy circular disc with a concentric circular hole, resting vertically on a horizontal plane. The problem is then modified, by applying suitable surface-tractions on the inner boundary, so as to apply to the case of a wheel to which a load is applied by an axle fitting into it.

1. *Vertical disc resting on two smooth pegs.*

We take the centre O of the disc, as origin, and the vertically upward direction, as the axis of x. Let A and B be two pegs at the same horizontal level, and let OA, OB make the same angle $\pi-\alpha$ with the vertical Ox, on either side of it.



Assuming the disc to be in a state of generalised plane stress, the stress equations of equilibrium are

$$\frac{\partial X_x}{\partial x}+\frac{\partial X_y}{\partial y}-w=0, \quad \frac{\partial X_y}{\partial x}+\frac{\partial Y_y}{\partial y}=0,$$

where w is the weight of the disc per unit area.

* "The stresses in a circular ring," Selected Engineering papers of the Institution of Civil Engineers, No. 12 (1924).

** "On stresses in circular rings under the action of isolated forces on the rim," Bull. Cal. Math. Soc., xxii, 27-38.

These equations are satisfied by

$$\left.\begin{aligned} X_x &= \frac{\partial^2\chi}{\partial y^2}+\tfrac{1}{2}wx, \quad Y_y = \frac{\partial^2\chi}{\partial x^2}-\tfrac{1}{2}wx, \\ X_y &= -\frac{\partial^2\chi}{\partial x\partial y}+\tfrac{1}{2}wy. \end{aligned}\right\} \tag{1}$$

Hence $$\nabla_1^2\chi = X_x+Y_y = 2(\lambda'+\mu)\Delta,$$

where λ' is the plane stress constant and is equal to $2\lambda\mu/(\lambda+2\mu)$.

It is seen from the equations of equilibrium, in terms of the displacements, that

$$\nabla_1^2\Delta = 0,$$

so that $$\nabla_1^4\chi = 0. \tag{2}$$

Now the stress system (1) can be divided into two parts, the first due to the stress function χ and the second due to the terms containing w. When expressed in polar co-ordinates, they are respectively

$$\left.\begin{aligned} \widehat{rr} &= \frac{1}{r^2}\frac{\partial^2\chi}{\partial\theta^2}+\frac{1}{r}\frac{\partial\chi}{\partial r}, \\ \widehat{\theta\theta} &= \frac{\partial^2\chi}{\partial r^2}, \quad \widehat{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\chi}{\partial\theta}\right), \end{aligned}\right\}. \tag{3}$$

and

$$\left.\begin{aligned} \widehat{rr} &= \tfrac{1}{2}wr\cos\theta, \quad \widehat{\theta\theta} = -\tfrac{1}{2}wr\cos\theta, \\ \widehat{r\theta} &= -\tfrac{1}{2}wr\sin\theta. \end{aligned}\right\} \tag{4}$$

The disc is acted upon by two equal forces P at A, B along AO, BO respectively. We have then for the equilibrium of the disc,

$$2P\cos\alpha = \pi a^2 w, \tag{5}$$

where a is the radius of the disc.

Let χ_1 be the stress function due to the force P at A.

Then

$$\chi_1 = -\frac{P}{\pi}r_1\theta_1\sin\theta_1, \tag{6}$$

where r_1, θ_1 are the polar coordinates of any point of the disc, referred to A as pole and AO as the initial line. Let this point of the disc have polar coordinates r, θ, when O is taken as the pole and Ox as the initial line.

The stress function χ_1 gives rise to the stress system

$$\widehat{r_1 r_1} = -\frac{2P}{\pi} \cdot \frac{\cos\theta_1}{r_1}, \quad \widehat{r_1\theta_1} = 0, \quad \widehat{\theta_1\theta_1} = 0.$$

At any point P on the boundary of the disc,

$$r_1 = 2a\cos\theta_1, \quad 2\theta_1 = \theta + \alpha,$$

and the direction cosines of the directions of r, θ with respect to those of r_1, θ_1 are respectively

$$(\cos\theta, \sin\theta), \ (-\sin\theta, \cos\theta).$$

Hence in (r, θ) coordinates, the stress system at a point P on the rim, due to (6), is

$$\left.\begin{aligned} \widehat{rr} &= -\frac{P}{2\pi a}\{1+\cos(\theta+\alpha)\}, \\ \widehat{\theta\theta} &= -\frac{P}{2\pi a}\{1-\cos(\theta+\alpha)\}, \\ \widehat{r\theta} &= \frac{P}{2\pi a}\sin(\theta+\alpha). \end{aligned}\right\} \tag{7}$$

The stress function χ_2 due to the force P at B, is

$$\chi_2 = -\frac{P}{\pi} r_2\theta_2 \sin\theta_2, \tag{8}$$

where r_2, θ_2 are the polar coordinates of a point of the disc, when B is taken as the pole and BO as the initial line.

As before, we calculate from (8), the stresses in (r, θ) coordinates, at a point P of the rim and we obtain

$$\left.\begin{aligned} \widehat{rr} &= -\frac{P}{2\pi a}\{1+\cos(\theta-\alpha)\}, \\ \widehat{\theta\theta} &= -\frac{P}{2\pi a}\{1-\cos(\theta-\alpha)\}, \\ \widehat{r\theta} &= \frac{P}{2\pi a}\sin(\theta-\alpha). \end{aligned}\right\} \tag{9}$$

Adding (4), (7) and (9), we have at a point P of the rim,

$$\widehat{rr} = \tfrac{1}{2} wa \cos\theta - \frac{P}{\pi a}(1+\cos\theta\cos\alpha),$$

$$\widehat{\theta\theta} = -\tfrac{1}{2} wa \cos\theta - \frac{P}{\pi a}(1-\cos\theta\cos\alpha),$$

$$\widehat{r\theta} = -\tfrac{1}{2} wa \sin\theta + \frac{P}{\pi a}\sin\theta\cos\alpha,$$

which reduce to

$$\widehat{rr} = -\frac{P}{\pi a}, \quad \widehat{\theta\theta} = -\frac{P}{\pi a}, \quad \widehat{r\theta} = 0, \tag{10}$$

if we take account of the relation (5).

Let us now introduce a stress function

$$\chi_3 = Ar^2, \tag{11}$$

which gives

$$\widehat{rr} = 2A, \quad \widehat{r\theta} = 0,$$

so that, if we choose $A = P/2\pi a$, the boundary stress disappears, except at the points A and B.

Therefore

$$\chi = \chi_1 + \chi_2 + \chi_3,$$

satisfies all the conditions of the problem.

If the pegs A and B are not in the same horizontal level, we take $\angle\, AOx = \pi - \alpha$ and $\angle\, BOx = \pi - \beta$. Here the reactions of the pegs are no longer equal. Let them be P_1 and P_2 respectively. Then, for equilibrium, we must have

$$\left.\begin{aligned} P_1\cos\alpha + P_2\cos\beta &= \pi a^2 w, \\ P_1\sin\alpha &= P_2\sin\beta, \end{aligned}\right\} \tag{12}$$

which give

$$P_1 = \frac{\pi a^2 w \operatorname{cosec}\alpha}{\cot\alpha + \cot\beta}, \quad P_2 = \frac{\pi a^2 w \operatorname{cosec}\beta}{\cot\alpha+\cot\beta}. \tag{13}$$

We then take

$$\chi_1 = -\frac{P_1}{\pi} r_1\theta_1 \sin\theta_1. \tag{14}$$

which gives as before

$$\left.\begin{aligned}\widehat{rr} &= -\frac{P_1}{2\pi a}\{1+\cos(\theta+\alpha)\},\\ \widehat{r\theta} &= \frac{P_1}{2\pi a}\sin(\theta+\alpha),\end{aligned}\right\} \tag{15}$$

at a point on the rim of the disc.

Also

$$\chi_2 = -\frac{P_2}{\pi} r_2\theta_2 \sin\theta_2, \tag{16}$$

which gives at a point on the rim,

$$\left.\begin{aligned}\widehat{rr} &= -\frac{P_2}{2\pi a}\{1+\cos(\theta-\beta)\},\\ \widehat{r\theta} &= \frac{P_2}{2\pi a}\sin(\theta-\beta).\end{aligned}\right\} \tag{17}$$

Hence, from (4), (15) and (17), we have at a point on the rim of the disc,

$$\widehat{rr} = \tfrac{1}{2}wa\cos\theta - \frac{1}{2\pi a}\{(P_1+P_2)+P_1\cos(\theta+\alpha)+P_2\cos(\theta-\beta)\},$$

$$\widehat{r\theta} = -\tfrac{1}{2}wa\sin\theta + \frac{1}{2\pi a}\{P_1\sin(\theta+\alpha)+P_2\sin(\theta-\beta)\},$$

and these reduce to

$$\widehat{rr} = -\frac{P_1+P_2}{2\pi a}, \quad \widehat{r\theta} = 0, \tag{18}$$

if we take account of (12).

Therefore, taking

$$\chi_3 = \frac{P_1+P_2}{4\pi a} r^2, \tag{19}$$

we see that

$$\chi = \chi_1+\chi_2+\chi_3$$

satisfies all the conditions of the problem.

2. *Vertical disc with a circular hole, resting on a horizontal plane.*

The stress system is given, as in § 1, by (3) and (4).

The disc is now acted upon by a vertically upward force $\pi(a^2-b^2)w$, at its point of contact with the horizontal plane, where b

is the inner radius of the disc. If r_1, θ_1 be the polar coordinates of a point of the disc, when the point of contact is taken as the pole and the vertically upward direction as the initial line, the stress function due to this force is

$$\chi_1 = -(a^2-b^2)\,wr_1\theta_1 \sin\theta_1, \tag{20}$$

so that at a point on the outer rim $r = a$,

$$\widehat{rr} = -\frac{(a^2-b^2)w}{2a}(1+\cos\theta), \quad \widehat{r\theta} = \frac{(a^2-b^2)w}{2a}\sin\theta. \tag{21}$$

Adding (4) and (21), we have at a point of the rim $r = a$,

$$\widehat{rr} = -\frac{(a^2-b^2)w}{2a} + \frac{b^2w}{2a}\cos\theta, \quad \widehat{r\theta} = -\frac{b^2w}{2a}\sin\theta. \tag{22}$$

To calculate the stresses across the inner boundary $r = b$, from the stress function χ_1, we transform it into (r, θ) coordinates, and we have

$$\chi_1 = -(a^2-b^2)\,wr\sin\theta\tan^{-1}\frac{r\sin\theta}{a+r\cos\theta}.$$

But when $\frac{r}{a} < 1$,

$$\tan^{-1}\frac{r\sin\theta}{a+r\cos\theta} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\frac{r}{a}\right)^n \sin n\theta,$$

so that

$$\chi_1 = -\tfrac{1}{2}a(a^2-b^2)w\left[\left(\frac{r}{a}\right)^2 - \tfrac{1}{2}\left(\frac{r}{a}\right)^3\cos\theta\right.$$

$$\left. + \sum_{n=2}^{\infty}(-1)^{n+1}\left\{\frac{1}{n-1}\left(\frac{r}{a}\right)^n - \frac{1}{n+1}\left(\frac{r}{a}\right)^{n+2}\right\}\cos n\theta\right]. \tag{23}$$

Therefore, when $r = b$, we have

$$\widehat{rr} = -\frac{(a^2-b^2)w}{2a}\left[2 - \frac{b}{a}\cos\theta\right.$$

$$\left. + \sum_{n=2}^{\infty}(-1)^n\left\{n\left(\frac{b}{a}\right)^{n-2} - (n-2)\left(\frac{b}{a}\right)^n\right\}\cos n\theta\right],$$

$$\widehat{r\theta} = \frac{(a^2-b^2)w}{2a}\left[\frac{b}{a}\sin\theta\right.$$

$$\left. + \sum_{n=2}^{\infty}(-1)^n\, n\left\{\left(\frac{b}{a}\right)^{n-2} - \left(\frac{b}{a}\right)^n\right\}\sin n\theta\right].$$

Combining the stress system (4) with that due to (23), we have on $r = b$,

$$\left.\begin{aligned} \widehat{rr} &= \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos n\theta, \\ \widehat{r\theta} &= \sum_{n=1}^{\infty} \beta_n \sin n\theta, \end{aligned}\right\} \tag{24}$$

where

$$\alpha_0 = -\frac{(a^2-b^2)w}{a}, \quad \alpha_1 = \frac{b(2a^2-b^2)}{2a^2}w,$$

$$\alpha_n = (-1)^{n+1}. \frac{(a^2-b^2)w}{2a}\left\{n\left(\frac{b}{a}\right)^{n-2} - (n-2)\left(\frac{b}{a}\right)^n\right\},$$

$$\beta_1 = -\frac{b^3 w}{2a^2},$$

$$\beta_n = (-1)^n . \frac{(a^2-b^2)w}{2a} . n\left\{\left(\frac{b}{a}\right)^{n-2} - \left(\frac{b}{a}\right)^n\right\}.$$

To neutralise the stress (22) on $r = a$, and (24) on $r = b$, we take

$$\chi_2 = A_0 r^2 + B_0 \log r$$
$$+\left(A_1 r^3 + \frac{B_1}{r}\right)\cos\theta + C_1(r\theta\sin\theta - \nu r\log r\cos\theta)$$
$$+\sum_{n=2}^{\infty}\left\{A_n r^n + \frac{B_n}{r^n} + C_n r^{n+2} + \frac{D_n}{r^{n-2}}\right\}\cos n\theta, \tag{25}$$

where $\nu = \mu/(\lambda'+2\mu)$, λ' being the plane stress constant.

The stresses calculated from (25), are

$$\widehat{rr} = 2A_0 + \frac{B_0}{r^2}$$
$$+\left\{2A_1 r - \frac{2B_1}{r^3} + \frac{(2-\nu)C_1}{r}\right\}\cos\theta$$
$$-\sum_{n=2}^{\infty}\left\{n(n-1)A_n r^{n-2} + \frac{n(n+1)B_n}{r^{n+2}}\right.$$
$$\left. + (n+1)(n-2)C_n r^n + \frac{(n-1)(n+2)D_n}{r^n}\right\}\cos n\theta, \tag{26a}$$

and
$$\widehat{r\theta} = \left\{ 2A_1 r - \frac{2B_1}{r^3} - \frac{\nu C_1}{r} \right\} \sin\theta$$
$$+ \sum_{n=2}^{\infty} \left\{ n(n-1)\, A_n r^{n-2} - \frac{n(n+1)B_n}{r^{n+2}} + n(n+1)C_n r^n - \frac{n(n-1)\, D_n}{r^n} \right\} \sin n\theta. \qquad (26b)$$

Now, let us choose the constants in χ_2, in such a way that
$$\left.\begin{aligned} \widehat{rr} &= \frac{(a^2-b^2)w}{2a} - \frac{b^2 w}{2a}\cos\theta, \\ \widehat{r\theta} &= \frac{b^2 w}{2a}\sin\theta, \end{aligned}\right\} \qquad (27)$$
when $r = a$, and
$$\left.\begin{aligned} \widehat{rr} &= -a_0 - \sum_{n=1}^{\infty} a_n \cos n\theta, \\ \widehat{r\theta} &= -\sum_{n=1}^{\infty} \beta_n \sin n\theta, \end{aligned}\right\} \qquad (28)$$
when $r = b$.

Then the stress function
$$\chi = \chi_1 + \chi_2$$
is appropriate for the problem, since it gives zero stresses on the inner boundary $r = b$, and also zero stresses on the outer boundary $r = a$, except at the point of contact with the ground.

In this way, we find a sufficient number of equations, for the determination of the constants ; only we get four equations for the three constants A_1, B_1 and C_1, and we find that these equations are consistent with one another.

The constants, thus calculated, are found to be
$$A_0 = \tfrac{1}{4} aw + \tfrac{1}{2} \cdot \frac{a_0 b^2}{a^2-b^2} = \tfrac{1}{4} aw - \frac{b^2 w}{2a},$$
$$B_0 = -\tfrac{1}{2} ab^2 w - \frac{a_0 a^2 b^2}{a^2-b^2} = \tfrac{1}{2} ab^2 w,$$
$$A_1 = \frac{b^2 w}{4a^2} - \frac{\nu b^2 w}{4(a^2+b^2)},$$
$$B_1 = \frac{\nu a^2 b^4 w}{4(a^2+b^2)},$$
$$C_1 = -\tfrac{1}{2} b^2 w,$$

and for $n \geq 2$,

$$A_n = -\frac{(n-1)\ P_n a^2 + Q_n a^{-2n+2}}{2n(n-1)R_n},$$

$$B_n = \frac{P_n a^{2n+2} - (n+1)\ Q_n a^2}{2n(n+1)R_n},$$

$$C_n = \frac{P_n}{2(n+1)R_n},$$

$$D_n = \frac{Q_n}{2(n-1)R_n},$$

where

$$P_n = (n+1)(\alpha_n + \beta_n) b^{-n+2}\ (a^2 - b^2) + (\beta_n - \alpha_n) b^{n+2} (a^{-2n+2} - b^{-2n+2}),$$

$$Q_n = (\alpha_n + \beta_n)\ b^{-n+2} (a^{2n+2} - b^{2n+2}) - (n-1)\ (\beta_n - \alpha_n)\ b^{n+2} (a^2 - b^2),$$

$$R_n = (n^2 - 1)\ (a^2 - b^2)^2 + (a^{2n+2} - b^{2n+2})\ (a^{-2n+2} - b^{-2n+2}).$$

Let us now modify the problem, by assuming that the inner boundary, instead of being free from tractions, is acted upon by a normal traction, $p \cos \theta$, so that the resultant force on the inner boundary is vertically downwards and is equal to

$$\int_{-\pi}^{\pi} p \cos^2\theta .\ b d\theta = \pi b p.$$

This corresponds to the case of a load applied to the disc, by an axle which is rigidly fixed to the disc. In this case, the reaction of the ground is equal to $\pi(a^2 - b^2)w + \pi b p$, so that there is no traction on the outer rim except at the point of contact, where there is a vertically upward force $\pi(a^2 - b^2)w + \pi b p$.

We then replace w by w' in the equations (20), (21) and (23) and replace (22) by

$$\left.\begin{aligned} \widehat{rr} &= -\frac{(a^2 - b^2)w'}{2a} + \frac{b(bw - p)}{2a} \cos\theta, \\ \widehat{r\theta} &= \qquad\qquad\qquad -\frac{b(bw - p)}{2a} \sin\theta, \end{aligned}\right\} \qquad (22')$$

where $w' = w + bp/(a^2 - b^2)$.

In (24), α_0, α_n, β_n have the same values as before, except that w is replaced by w'; α_1, β_1 have now the values

$$\alpha_1 = \frac{b(2a^2-b^2)w}{2a^2} + \frac{b^2p}{2a^2},$$

$$\beta_1 = -\frac{b^3w}{2a^2} + \frac{b^2p}{2a^2}.$$

We keep the equations (25), (26a) and (26b) as they are, and replace (27) and (28) respectively by

$$\left.\begin{aligned} \widehat{rr} &= \frac{(a^2-b^2)w'}{2a} - \frac{b(bw-p)}{2a}\cos\theta, \\ \widehat{r\theta} &= \frac{b(bw-p)}{2a}\sin\theta, \end{aligned}\right\} \quad (27')$$

when $r = a$, and

$$\left.\begin{aligned} \widehat{rr} &= -\alpha_0 - (\alpha_1 - p)\cos\theta - \sum_{n=2}^{\infty} \alpha_n \cos n\theta, \\ \widehat{r\theta} &= -\sum_{n=1}^{\infty} \beta_n \sin n\theta, \end{aligned}\right\} \quad (28')$$

when $r = b$.

The constants A_0, B_0, A_n, B_n, C_n, D_n have the same values as before, except that w is replaced by w'; and A_1, B_1, C_1 are given by

$$A_1 = \frac{b(bw-p)}{4a^2} - \frac{\nu b(bw-p)}{4(a^2+b^2)},$$

$$B_1 = \frac{\nu\, a^2b^3(bw-p)}{4(a^2+b^2)},$$

$$C_1 = -\tfrac{1}{2}\, b(bw-p).$$

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Sur l'équation aux dérivées partielles qui se présente dans la théorie de la propagation de l'électricité

By

Maurice de Duffahel.

Les variations du potentiel électrique dans un fil qui transmet une perturbation électrique ont été, comme l'on sait, représentées par l'équation

$$A \frac{\partial^2 V}{\partial t^2} + 2B \frac{\partial V}{\partial t} = C \frac{\partial^2 V}{\partial x^2} ;$$

V est le potentiel, et A, B, C, sont trois constantes positives.

En choisissant convenablement les unités, on peut réduire l'équation à la forme

$$\frac{\partial^2 V}{\partial t^2} + 2\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2},$$

et enfin, en posant $\qquad V = U^{-t},$

on a l'équation $\qquad \dfrac{\partial^2 U}{\partial t^2} - \dfrac{\partial^2 U}{\partial x^2} = U.$

H. Poincaré a fait, dans les *Comptes rendus*, * une discussion des intégrales de cette équation, fort intéressante au point de vue physique; je veux montrer comment l'équation précédente pourrait être discutée de la manière la plus simple et la plus rigoureuse à l'aide de la méthode générale de Riemann, méthode fondamentale dans la théorie des équations aux dérivées partielles du second ordre à caractéristiques réelles : c'est ce que je me propose de développer ici.

1. Je rappellerai d'abord les résultats de Riemann, en me reportant à la remarquable exposition faite par G. Darboux, dans le tome II de ses *Leçons sur la théorie générale des surfaces* (p. 75). Prenons l'équation

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0,$$

* H. Poincaré, *Sur la propagation de l'électricité* (*Comptes rendus*, 26 décembre 1893).

et soit considérée en même temps son adjointe

$$\frac{\partial^2 u}{\partial x \partial y} - a\frac{\partial u}{\partial x} - b\frac{\partial u}{\partial y} + \left(c - \frac{\partial a}{\partial x} - \frac{\partial b}{\partial y}\right) u = 0,$$

et posons

$$M = auz + \tfrac{1}{2}\left(u\frac{\partial z}{\partial y} - z\frac{\partial u}{\partial y}\right),$$

$$N = buz + \tfrac{1}{2}\left(u\frac{\partial z}{\partial x} - z\frac{\partial u}{\partial x}\right).$$

Soient B'C' (Fig. 1), une courbe tracée arbitrairement dans le plan, et A un point quelconque de ce plan.

En désignant par x_0, y_0, les coordonnées de A, supposons que l'on ait déterminé la solution

$$u\,(x, y; x_0, y_0)$$

de l'équation adjointe, se réduisant à l'unité pour $x = x_0$, $y = y_0$, et prenant la valeur

$$e^{\int_{x_0}^{x} b\,dx}$$

pour $y = y_0$, tandis qu'elle prend la valeur

$$e^{\int_{y_0}^{y} a\,dy}$$

pour $x = x_0$. Dans ces conditions, on aura

$$(uz)_A = \frac{(uz)_B + (uz)_G}{2} - \int_G^B (N\,dx - M\,dy),$$

l'intégrale curviligne étant prise le long de C'B'.

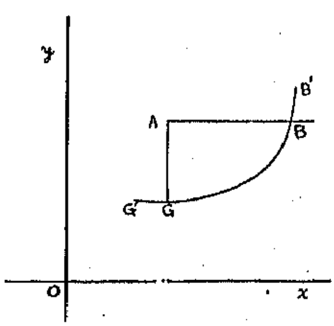

Fig. 1.

Cette formule permet de déterminer la solution z de l'équation aux dérivées partielles proposée, qui prend des valeurs données ainsi que l'une de ses deux dérivées pour les points de la courbe B'C'. L'équation

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy,$$

appliquée à un déplacement suivant cette courbe, détermine évidement celle des deux dérivées premières qui n'est pas donnée *a priori.*

2. Ceci rappelé, revenons à l'équation

$$\frac{\partial^2 z}{\partial t^2} - \frac{\partial^2 z}{\partial x^2} = z, \tag{1}$$

en remplaçant U par z. Transformons-la en posant

$$2X = x + t, \qquad 2Y = x - t.$$

On obtient tout de suite

$$\frac{\partial^2 z}{\partial X \partial Y} + z = 0. \tag{2}$$

L'équation adjointe est ici :

$$\frac{\partial^2 u}{\partial X \partial Y} + u = 0,$$

c'est la même équation.

Supposons qu'une intégrale de l'équation (1) soit déterminée par les conditions initiales suivantes : on se donne z et $\frac{\partial z}{\partial t}$ pour $t = 0$, ces valeurs étant seulement *différentes de zéro* pour x compris a et b $(a > b)$.

On voit alors que, pour l'équation (2), on peut considérer la fonction z ainsi que ses dérivées partielles du premier ordre *comme données sur la bissectrice de l'angle des axes*; les valeurs données sont seulement *différentes de zéro* sur un segment fini de cette bissectrice, à savoir le segment déterminé par les équations

$$X = Y = \frac{x}{2},$$

x variant de b à a.

3. Pour appliquer la méthode de Riemann, nous avons seulement à voir si nous pouvons déterminer l'intégrale u de l'équation adjointe

$$\frac{\partial^2 u}{\partial X \partial Y} + u = 0,$$

qui pour $X = X_0$ prend la valeur *un* quelque soit Y, et qui pour $Y = Y_0$ prend la valeur *un* quelque soit X. Or cette recherche est immédiate, car, en posant

$$\lambda = (X - X_0)(Y - Y_0),$$

on trouvera une fonction $u = \phi(\lambda)$, satisfaisant à l'équation précédente; en effet, on obtient ainsi l'équation de Bessel

$$\lambda \frac{d^2\phi}{d\lambda^2} + \frac{d\phi}{d\lambda} + \phi = 0,$$

et la fonction cherchée est la série de Bessel

$$u = 1 - \frac{\lambda}{1^2} + \frac{\lambda^2}{(1.2)^2} - \frac{\lambda^3}{(1.2.3)^2} + \ \ldots\ldots\ldots$$

4. Figurons (*fig.* 2) le plan (X, Y).

L'intégrale z est nulle ainsi que $\frac{\partial z}{\partial X}$ $\left(\text{et par suite } \frac{\partial z}{\partial Y}\right)$ sur la bissectrice des axes, sauf sur la partie $\beta\alpha$; on a sur cette partie des successions de valeurs données pour z et $\frac{\partial z}{\partial X}$ $\left(\text{d'où résulte la valeur de } \frac{\partial z}{\partial Y}\right)$. Nous avons donc la formule

$$z_A = \frac{(uz)_B + (uz)_G}{2} - \frac{1}{2}\int_G^B \left[\left(u\frac{\partial z}{\partial X} - z\frac{\partial u}{\partial X}\right)dX - \left(u\frac{\partial z}{\partial Y} - z\frac{\partial u}{\partial Y}\right)dY\right],$$

ce que l'on peut écrire :

$$z_A = \frac{(uz)_B + (uz)_G}{2} + \frac{1}{2}\int_G^B \phi\left(\frac{\partial z}{\partial X} - \frac{\partial z}{\partial Y}\right)dX - \frac{(X_0 - Y_0)}{2}\int_G^B z\frac{\partial \phi}{\partial \lambda}\,dX.$$

On peut regarder sous les signes d'intégration $\frac{\partial z}{\partial X} - \frac{\partial z}{\partial Y}$ et z comme des fonctions arbitrairement données de X, différentes de zéro entre $\frac{a}{2}$ et $\frac{b}{2}$ et nulles pour toute autre valeur de X; désignons les

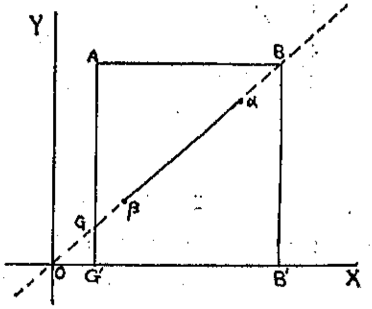

FIG. 2.

par $\psi(X)$ et $\chi(X)$. Nous aurons donc

$$z(t_0, x_0) = \frac{(uz)_B + (uz)_G}{2} - \frac{1}{2}\int_{G'}^{B'} \phi(\lambda)\psi(X)dX$$

$$- t_0 \int_{G'}^{B'} \chi(X)\phi'(\lambda)dX,$$

$$2X_0 = x_0 + t_0, \qquad 2Y_0 = x_0 - t_0,$$

en désignant pas G' et B' les projections de C et B sur OX, et en posant

$$\lambda = \left(X - \frac{x_0 + t_0}{2}\right)\left(Y - \frac{x_0 - t_0}{2}\right).$$

Telle est la forme de l'intégrale générale de l'équation (1).

5. Discutons la valeur de $z(t_0, x_0)$, quand, x_0 étant arbitraire mais fixe, t_0 varie de 0 *à* $+\infty$.

Supposons x_0 en dehors de l'intervalle ba. Les relations

$$2X_0 = x_0 + t_0, \qquad 2Y_0 = x_0 - t_0,$$

montrent que le point A de coordonnées X_0, Y_0 se déplace sur une droite perpendiculaire à $\alpha\beta$, comme l'indique la figure 3 qui correspond à $x_0 > a$.

Tant que t_0 est trop petit pour que le point B soit situé dans

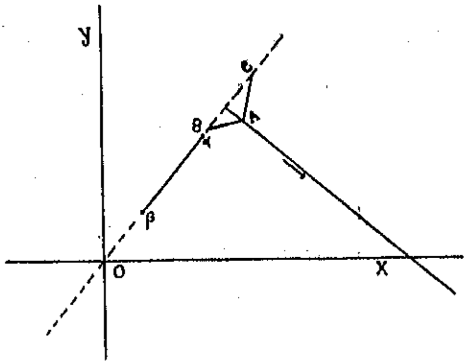

FIG 3.

l'intervalle $\alpha\beta$, tous les termes figurant dans l'expression de $z(t_0, x_0)$ sont nuls. C'est seulement quand B arrive en α que z devient différent de zéro; ceci se produit quand

$$\frac{x_0 - t_0}{2} = \frac{a}{2},$$

c'est-à-dire à l'instant $x_0 - a$, et la partie ne contenant pas de signe d'intégration dans $z(t_0, x_0)$, c'est-à-dire $\frac{(uz)_B}{2}$, sera *différente de zéro* depuis le temps $x_0 - a$ jus-qu'an temps $x_0 - b$. On peut dire que *cette partie correspond à une onde régulière.*

Une fois le temps $x_0 - b$ dépassé, la valeur de z se réduit à

$$z(t_0, x_0) = -\frac{1}{2}\int_{\frac{b}{2}}^{\frac{a}{2}} \phi(\lambda)\psi(X)dX - t_0\int_{\frac{b}{2}}^{\frac{a}{2}} \chi(X)\phi'(\lambda)dX.$$

On voit donc ainsi bien nettement *une sorte de résidu laissé par l'onde régulière que représentait le terme* $\frac{(uz)_B}{2}$.

Quant à la valeur asymptotique de $z(t_0, x_0)$ ou plutôt de

$$V(t_0, x_0) = e^{-t_0} z(t_0, x_0)$$

pour t_0 très grand, elle serait facile à obtenir à l'aide de l'expression asymptotique bien connue de $\phi(\lambda)$ pour λ infini, mais je ne m'y arrêterai pas; ce que nous venons de dire suffit pour montrer la nature des intégrales de *l'équation des télégraphistes*.

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20

Sur les Couples de Fonctions Uniformes d'une Variable.

(Remarques se rapportant aux points d'une courbe algébrique de genre supérieur à l'unité.)

By

Maurice de Duffahel.

1. J'ai établi autrefois la proposition suivante.

Si entre deux fonctions analytiques uniformes d'une variable existe une relation algébrique de genre supérieur à l'unité, ces fonctions ne peuvent avoir de point singulier essentiel isolé.

Je me suis servi, pour la démonstration,[1] d'une transcendante de la théorie des fonctions fuchsiennes, telle qu'elle a été constituée par H. Poincaré. Comme je le faisais alors remarquer, ce théorème montre la nécessité de l'introduction des fonctions automorphes ou de fonctions analogues dans la théorie des courbes algébriques, puisque, en laissant de côté les genres *zéro* et *un*, il est impossible d'obtenir une représentation paramétrique uniforme par des fonctions ayant des points singuliers essentiels *isolés*. J'ajoute, ce qui n'est pas sans quelque intérêt historique, que c'est en s'appuyant sur le théorème précédent que H. Poincaré donna le premier exemple[2] de fonctions uniformes, sans lignes singulières, mais avec des points singuliers formant un ensemble parfait : ce sont les fonctions fuchsiennes, de genre supérieur à *un*, existant dans tout le plan.

2. Les considérations employées dans la démonstration du théorème précédent peuvent être utilisées dans diverses questions *concernant les courbes de genre supérieur à un*, de façon à obtenir des propositions présentant une grande analogie et aussi des différences

La démonstration suppose seulement que les fonctions ont une branche uniforme dans la voisinage d'un point singulier essentiel isolé, sans qu'intervienne la façon dont elles se comportent ailleurs.

[2] Voir la *Notice sur les travaux scientifiques de H. Poincaré*, 18 (Paris, Gauthier-Villars, 1886).

sensibles avec des résultats obtenus dans ces dernières années pour une seule fonction uniforme. Je vais en donner quelques exemples dans ce petit Mémoire.

3. On sait que M. Landau a fait connaître une extension extrêmement remarquable, aujourd'hui classique, du premier théorème qu'il a donné jadis sur les fonctions entières. Cette généralisation consiste en ce que, si l'on a un développement taylorien commençant par les termes

$$a_0 + a_1 z + \dots\dots\dots, \quad (a_1 \neq 0)$$

convergent dans le cercle de rayon R, et si l'on suppose que dans ce cercle la fonction ainsi définie ne devienne jamais égale à *zéro* et *un*, le rayon R est au plus égal à une certaine fonction

$$R(a_0, a_1),$$

dépendant uniquement de a_0 et de a_1. Carathéodory a fait connaître une limite précise pour cette dernière fonction. Cette question et d'autres analogues ont été présentées avec une rare élégance et sous un jour nouveau par Lindelöf dans une communication faite au congrès des mathématiciens scandinaves.

Ceci rappelé, nous partons de la courbe algébrique

$$f(x, y) = 0, \tag{1}$$

de genre au moins égal à *deux*. Il résulte de la théorie des fonctions fuchsiennes qu'on peut former une fonction $\lambda(x, y)$ du point analytique (x, y), holomorphe dans le voisinage de tout point de la surface de Riemann correspondant à (1), et pour laquelle le coefficient de i est toujours positif. Les diverses déterminations de $\lambda(x, y)$ se déduisent d'ailleurs de l'une d'elles par des substitutions linéaires, et de plus l'inversion de λ conduit à exprimer x et y par des fonctions automorphes.

Admettons maintenant qu'on puisse satisfaire à l'équation (1) par des fonctions x et y d'une variable z, *méromorphes* à l'intérieur du cercle C de rayon R ayant l'origine pour centre; on suppose de plus que, pour $z = 0$, on ait $x = a$, $y = b$, le point (a, b) étant un point ordinaire déterminé de la courbe. Enfin, soient les développements tayloriens de x et y autour de $z = 0$:

$$\begin{cases} x = a + a_1 z + \dots\dots\dots, \\ y = b + b_1 z + \dots\dots\dots, \end{cases} \quad (a_1 \neq 0). \tag{2}$$

où il est manifeste que b_1 s'exprime à l'aide de a et de a_1.

Nous substituons dans la fonction $\lambda(x, y)$ à la place de (x, y) les fonctions méromorphes de z, dont il vient d'être parlé. La fonction λ devient alors une fonction de z, holomorphe dans le cercle C, et le coefficient de i dans cette fonction est positif. Considérons alors l'expression

$$\mathrm{E}(z) = \frac{\lambda(x, y)-\lambda(a, b)}{\lambda(x, y)-\lambda_0(a, b)},$$

en désignant par $\lambda_{\bar{0}}(a, b)$ la quantité imaginaire conjugée de $\lambda(a, b)$. Le module de la fonction $\mathrm{E}(z)$ holomorphe dans le cercle C est inférieur à l'unité. Développons cette fonction suivant les puissances de z. Commençons à cet effet par développer $\lambda(x, y)$ suivant les puissances de $x-a$ dans le voisinage du point analytique (a, b); on a ainsi

$$\lambda(x, y) = \mu(a)+(x-a)\mu'(a)+\ldots\ldots,$$

$\mu(x)$ étant holomorphe dans le voisinage de a. Nous avons alors

$$\mathrm{E}(z) = \mathrm{A}_1 z+\mathrm{A}_2 z^2+\ldots\ldots,$$

où

$$\mathrm{A}_1 = \frac{a_1\mu'(a)}{\mu(a)-\mu_0(a)},$$

$\mu_0(a)$ étant le conjugué de $\mu(a)$.

Puisque, dans le cercle de rayon R, le module $\mathrm{E}(z)$ est inférieur à *un*, on aura

$$|\mathrm{A}_1\mathrm{R}| \leqq 1,$$

et par suite

$$\mathrm{R} \leqq \left|\frac{\mu(a)-\mu_0(a)}{a_1\mu'(a)}\right|.$$

Nous avons donc le théorème suivant, où nous posons

$$\mathrm{R}(a, a_1) = \left|\frac{\mu(a)-\mu_0(a)}{a_1\mu'(a)}\right|.$$

Considéront un point (a, b) de la courbe f, on met à la place de x dans l'équation

(a) $$f(x, y) = 0,$$

une fonction méromorphe de z dans un certain domaine autour de l'origine, dont le développement taylorien autour de $z=0$ est donné par

la formule

$$z = a + a_1 z + \ldots\ldots$$

On tire de (α) *la fonction y de z, prenant z = 0 la valeur de b. Les deux fonctions x et y de z ne pourront être simultanément méromorphes dans un cercle ayant l'origine pour centre et un rayon supérieur à* $R(a, a_1)$ *expression qui dépend seulement de a et de* a_1, *et nullement des autres coefficients du développement de x.*

4. Le théorème précédent peut être généralisé en utilisant, au lieu de la fonction $\lambda(x, y)$, une autre fonction de même nature. On peut employer la fonction $\pi(x, y)$, jouissant de propriétés analogues à celles de $\lambda(x, y)$, sauf qu'elle admettra sur la surface de Riemann des points singuliers de nature logarithmique

$$(8) \qquad (\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots\ldots\ldots (\alpha_q, \beta_q),$$

l'entier q étant quelconque.

Considérons alors deux fonctions x et y d'une variable z, satisfaisant à (1), méromorphes à l'intérieur du cercle C de rayon R ayant l'origine pour centre, et telles que le point de (x, y) ne coincide dans ce cercle avec aucun des points (α_h, β_h) de la suite (8). Substituons alors dans la fonction $\pi(x, y)$, à la place de (x, y), les fonctions méromorphes dont il vient d'être parlé; la fonction π devient une fonction de z, holomorphe dans le cercle C, et le coefficient de i dans cette fonction est positif.

Il n'y a plus qu'à raisonner comme nous l'avons fait plus haut, pour savoir que *le rayon R est inférieur à une certaine fonction de a et de* a_1, en désignant toujours par

$$x = a + a_1 z + \ldots\ldots\ldots$$

le développement taylorien de x dans le voisinage de $z = 0$.

5. Indiquons une autre conséquence du même genre d'analyse dans un ordre d'idées différent. Divers géomètres ont indiqué d'importants théorèmes sur la convergence des suites de fonctions.[1] En laissant

[1] On peut voir pour la bibliographie un Mémoire de Carathéodory et Landau: Beiträge zur Konvergenz von Funktionenfolgen, Sitzungsberichte der Kgl. Preussischen Akademie der Wissenschaften (séance du 18 mai 1911, *587-618*). On consultera aussi, sur ce sujet, deux Notes importantes de M. P. Montel: "Sur les fonctions analytiques qui admettent deux valeurs exceptionnelles dans un domaine," Comptes rendus Ac. des Sc., tome 153, séance du 20 novembre 1911, *996-998*; "Sur l'indétermination d'une fonction uniforme dans le voisinage de ses points essentiels," (*ibid.*, séance du 26 décembre, *1455-1456*).

de côté le théorème élémentaire de Weierstrass, nous avons d'abord le théorème donné par Stieltjes dans son célèbre Mémoire sur les fonctions continues, [1] qui n'est d'ailleurs qu'un cas particulier d'un théorème plus récent de Vitali, susceptible d'être ainsi formulé : Soient les fonctions

$$f_1(z),\ f_2(z), \ldots\ldots\ldots f_n(z), \ldots\ldots\ldots$$

holomorphes dans le cercle C de rayon *un*, et telles que l'on ait, quelque soit n,

$$| f_n(z) | < g,$$

g étant un nombre fixe. On suppose que $f_n(z)$ ait une limite, n grandissant indéfiniment, pour une infinité de points de C ayant au moins un point de condensation à l'intérieur de C. *Dans ces conditions, $f_n(z)$ a une limite pour tous les points de l'intérieur du cercle, et cette limite est une fonction holomorphe de z à l'intérieur de C.*

Dans leur Mémoire (*loc. cit.*[1]) Carathéodory et Landau font connaître un théorème remarquable pour lequel l'énoncé est celui de Vitali, *sauf que la condition*

$$| f_n(z) | < g,$$

est remplacée par la condition qu'aucune des fonctions $f_n(z)$ ne prend dans le cercle C les valeurs a et b (a et b étant deux constantes distinctes).

6. Nous devons nous attendre à avoir, dans le domaine des courbes *de genre supérieur à m*, un théorème analogue à celui de Carathéodory et Landau, *mais où l'énoncé n'aura pas à envisager de valeurs exceptionnelles.*

Reprenons la courbe (1), soient

$$(x_1, y_1),\ (x_2, y_2), \ldots\ldots\ldots (x_n, y_n), \ldots\ldots\ldots$$

des couples de fonctions de la variable z, méromorphes dans un cercle C ayant l'origine pour centre, et satisfaisant aux équations $f(x_n, y_n) = 0$. Nous allons étudier le théorème suivant :

Si le couple (x_n, y_n) a une limite pour une infinité de points du cercle C, possédant au moins un point de condensation à son intérieur, le couple (x_n, y_n) aura une limite pour tous les points de l'intérieur de C, et les coordonnées de ce couple limite sont des fonctions méromorphes de z.

[1] Voir la correspondance d'Hermite et de Stieltjes (Paris, Gauthier-Villars), tome II, 1905, 368.

Il suffira de reprendre la fonction $\lambda(x, y)$ dont nous nous sommes servi plus haut, ou plutôt la combinaison déjà utilisée

$$\frac{\lambda(x, y)-\lambda(a, b)}{\lambda(x, y)-\lambda_0(a, b)},$$

en désignant par (a, b) un point déterminé de la courbe, et représentant par $\lambda_0(a, b)$ la quantité complexe de $\lambda(a, b)$.

La fonction de z

$$G_n(z) = \frac{\lambda(x_n, y_n)-\lambda(a, b)}{\lambda(x_n, y_n)-\lambda_0(a, b)},$$

est une fonction holomorphe dans le cercle C, et dont le module est inférieur à l'unité. Nous obtenons ainsi une suite de fonctions

$$G_1(z),\ G_2(z), \ldots\ldots\ldots G_n(z), \ldots\ldots\ldots$$

holomorphes dans C, et pour lesquelles

$$|G_n(z)| < 1.$$

On peut alors appliquer le théorème de Vitali, puisque $G_n(z)$, comme le couple (x_n, y_n), a une limite pour une infinité de points du cercle C possédant au moins un point de condensation dans C. De la limite de G_n se déduit de suite l'existence de la limite du couple (x_n, y_n), puisque l'inversion de λ se fait d'une manière uniforme. Le reste de démonstration est immédiat.

7. Les exemples précédents, qu'on pourrait multiplier, suffisent à montrer les analogies que présente avec le cas d'une seule fonction le cas d'un couple (x, y) de fonctions uniformes satisfaisant à f (de genre $p > 1$), et l'on voit aussi les différences. Celles-ci peuvent être brièvement formulées en disant que, dans le domaine des courbes de genre *zéro* et *un*, *on doit envisager des valeurs exceptionnelles*, tandis que, dans le domaine des courbes de genre supérieur à *un*, *il n'est pas nécessaire de considérer des valeurs exceptionnelles*. C'est de là que provient la différence des énoncés précédents avec ceux qui ont été rencontrés antérieurement.

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21

AN EXTENSION OF WILSON'S THEOREM

BY

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From the point of view of residues, Wilson's theorem,* *i.e.*,

$$(p-1)!+1 \overset{p}{\equiv} 0,$$

can be stated as follows:

$$\prod_{\substack{1 \leqq a < p \\ (a,p)=1}} a \overset{p}{\equiv} -1.$$

In this note, we prove a theorem of this character for any modulus m. It may be noted that the point of proof, is that the congruence

$$x^2 \overset{m}{\equiv} 1, \qquad (m \geqq 3),$$

has always an even number of solutions.

Theorem:

$$\prod_{\substack{1 \leqq a < m \\ (a,m)=1}} a \overset{m}{\equiv} \begin{cases} -1, \text{ if } m=4, \text{ or } p^l \text{ or } 2p^l, & (p > 2,\ l > 0), \\ +1, \text{ otherwise.} \end{cases}$$

Proof: The congruence

$$ax \overset{m}{\equiv} 1, \qquad \text{when } (a,m)=1, \tag{1}$$

has one and only one solution and it is obvious that $(x, m)=1$.

We now divide the $\phi(m)$ 'a's into two classes, (1) those for which $x \neq a$, and (2) those for which $x=a$. The former obviously divide themselves into pairs. The number n of the latter is the number of solutions of

$$x^2 \overset{m}{\equiv} 1. \tag{2}$$

* We write $a \overset{m}{\equiv} b$, when $m/(a-b)$; p denotes a prime.

It now follows from known theorems* that $n \overset{4}{\equiv} 2$, if $m=4$ or p^l or $2p^l$, where p is a prime >2 and $l>0$, and $n \overset{4}{\equiv} 0$, otherwise. Also if b is a solution of (2), then so is $m-b$ and $(b, m)=1$, $(m-b, m)=1$ and $m-b \neq b$. So, the 'a' s of class (2), can be divided into $\frac{1}{2}n$ pairs of the type b, $m-b$.

Hence

$$\prod_{\substack{1 \leq a < m \\ (a,m)=1}} a = \prod_{\substack{a^2 \overset{m}{\not\equiv} 1 \\ 1 \leq a < m \\ (a,m)=1}} a \cdot \prod_{\substack{a^2 \overset{m}{\equiv} 1 \\ 1 \leq a < m \\ (a,m)=1}} a \overset{m}{\equiv} \prod_{\substack{b \leq \frac{1}{2}m \\ b^2 \overset{m}{\equiv} 1}} b\,(m-b) \overset{m}{\equiv} (-1)^{\frac{n}{2}}$$

$$= \begin{cases} -1, \text{ if } m=4 \text{ or } p^l \text{ or } 2p^l, \ (p>2,\ l>0), \\ +1, \text{ otherwise.} \end{cases}$$

Remark: The result for $m=p^l$ and $2p^l$, also follows from the theory of primitive roots. This seems to suggest some connection between the result of the theorem and the theory of primitive λ roots,† although we fail to see what it is.

The preceding proof depends on the theory of quadratic residues. Since Wilson's theorem itself can be proved without recourse to the theory of quadratic residues or primitive roots, a proof of the theorem by induction alone, starting from Wilson's theorem, would be of some interest.

(i) Let $m=2^\lambda$, $\lambda \geq 1$.

Then

$$\left.\begin{aligned} \prod_{0<a<2} a &= 1 \overset{2}{\equiv} \pm 1 \\ \prod_{\substack{(a,4)=1 \\ 0<a<4}} a &= 1.3 \overset{4}{\equiv} -1 \end{aligned}\right\} \qquad (3)$$

and for $\lambda > 2$,

$$\prod_{\substack{(a,2)=1 \\ 0<a<2^\lambda}} a = \prod_{\substack{(a,2)=1 \\ 0<a<2^{\lambda-1}}} a \cdot \prod_{\substack{(a,2)=1 \\ 0<a<2^{\lambda-1}}} (2^\lambda - a) \overset{2^\lambda}{\equiv} (-1)^{2^{\lambda-2}} \prod_{\substack{(a,2)=1 \\ 0<a<2^{\lambda-1}}} a^2 \overset{2^\lambda}{\equiv} 1,$$

* See Landau, Vorlesungen über Zahlentheorie, Bd. 1, Satz 88 and Satz 71.

† See Carmichael, Theory of Numbers, Ch. 5.

if $\prod\limits_{\substack{(a,2)=1\\ o\leq a<2^{\lambda-1}}} a \overset{2^{\lambda-1}}{\equiv} \pm 1$; and the result follows from (3) by induction.

(*ii*) Let $m=p^l$, $p>2$, $l>0$.

Then, by Wilson's theorem,

$$\prod_{\substack{o<a<p\\(a,p)=1}} a = -1+\mu p, \tag{4}$$

and for $l>1$,

$$\prod_{\substack{o<a<p^l\\(a,p)=1}} a = \prod_{r=o}^{p-1} \prod_{\substack{o<a<p^{l-1}\\(a,p)=1}} (rp^{l-1}+a) \overset{p^l}{\equiv} \Big[\prod_{\substack{o<a<p^{l-1}\\(a,p)=1}} a\Big]^p$$

$$\Big(\text{since} \quad \sum_{\substack{o<a<p^{l-1}\\(a,p)=1}} 1/a \overset{p^{l-1}}{\equiv} o\Big)$$

$$=(-1+\nu p^{l-1})^p \overset{p^l}{\equiv} -1, \tag{5}$$

if $\prod\limits_{\substack{o<a<p^{l-1}\\(a,p)=1}} a \overset{p^{l-1}}{\equiv} -1$; and the result follows from (4) by induction.

Finally, let m be any composite number containing at least two distinct primes as factors. We now observe that for $a>1$, $b>1$, $(a,b)=1$, $(x,a)=1$, the congruence $y \overset{a}{\equiv} x$, $o\leq y<ab$, $(y,ab)=1$ has exactly $\phi(b)$ solutions.

(*iii*) Let $m=2p$, $p\geq 2$.

Then $\prod\limits_{\substack{o<a<2p\\(a,p)=1}} a \overset{p^\alpha}{\equiv} \prod\limits_{\substack{o<a<p^\alpha\\(a,p)=1}} a \overset{p^\alpha}{\equiv} -1.$

Hence

$$\prod_{\substack{o<a<2p^\alpha\\(a,p)=1}} a \overset{2p^\alpha}{\equiv} -1.$$

since the product of any number of odd integers $\overset{2}{\equiv} \pm 1$.

(*iv*) Let $m = p_1^{l_1}\ p_2^{l_2}\ \ldots\ p_r^{l_r}$ and $\neq 2p^l$, so that for every k,

$$\frac{m}{p_k^{l_k}} > 2 \text{ and consequently } \phi\left(\frac{m}{p_k^{l_k}}\right) \text{ is given.}$$

Then

$$\prod_{\substack{o<a<m \\ (a,m)=1}} a \overset{p_k^{l_k}}{\equiv} \Bigg[\prod_{\substack{o<a<p_k^{l_k} \\ (a,p_k)=1}} a\Bigg]^{\phi\left(\frac{m}{p_k^{l_k}}\right)} \overset{p_k^{l_k}}{\equiv} 1. \quad (k=1, 2, \ldots r)$$

Hence

$$\prod_{\substack{o<a<m \\ (a,m)=1}} a \overset{m}{\equiv} +1.$$

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22

Numerische Identitäten

By

Alfred Moessner

I. (*a*) Das Symbol $\overset{n}{=}$ bedeute, dass Summengleichheit in n verschiedenen Potenzen bestehe. So bedeutet zum Exempel

$$A, B, C, \overset{n}{=} D, E, F, G \qquad (n=1, 2, 3)$$

dass Gleichungssystem

$$A+B+C=D+E+F+G,$$

$$A^2+B^2+C^2=D^2+E^2+F^2+G^2,$$

$$A^3+B^3+C^3=D^3+E^3+F^3+G^3.$$

Nach einem Satze der Zahlentheorie folgt aus

$$A_1, A_2, A_3, \quad \ldots \overset{n}{=} B_1, B_2, B_3 \ldots$$

für $n=2, 4 \ldots 2k$ mit $k+1$ Gliedern auf jeder Seite die Identität

$$A_1-x,\ A_2-x,\ A_3-x,\ \ldots,\ B_1+x,\ B_2+x,\ B_3+x,\ \ldots$$

$$\overset{n}{=} B_1-x,\ B_2-x,\ B_3-x,\ \ldots,\ A_1+x,\ A_2+x,\ A_3+x,\ \ldots$$

für $n=1, 3, 5, \ldots 2k+1$ mit $2k+2$ Gliedern auf jeder Seite.

Dieser Satz sei in nachfolgenden Zeilen auf einige numerische Identitäten angewendet, wobei gewisse Bedingungen erfüllt werden sollen.

(*b*) Ist $k=1$, so gibt $n=2$ die Gleichung $A_1^2+A_2^2 = B_1^2+B_2^2$.

Hiebei sei $A_1 \neq A_2$ und $B_1 \neq B_2$. Wenden wir nun den Satz unter (a) auf diese Gleichung an, so erhalten wir Relationen von der Form

$$X_1, X_2, \quad \ldots \quad X_s \overset{n}{=} Y_1, Y_2 \quad \ldots \quad Y_v, \qquad (n=1,3),$$

wobei stets $(s+v) < 9$ ist und wobei alle Glieder positiv werden, wenn die negativen Glieder auf die entgegengesetzte Gleichungsseite geschafft werden. Wir bekommen die verschiedensten Relationen, je nachdem wir setzen

$$x = \frac{A_1+A_2}{2} \quad \text{oder} \quad x = \frac{B_1+B_2}{2} \quad \text{oder} \quad x = A_2-A_1$$

$$\text{oder} \quad x = B_2-B_1 \quad \text{oder} \quad x = \frac{A_1+B_1}{2} \quad \text{oder} \quad x = \frac{A_2+B_2}{2}$$

$$\text{oder} \quad x = \frac{A_1+B_2}{2} \quad \text{oder} \quad x = \frac{A_2+B_1}{2}.$$

Hier seien einige numerische Exempel angegeben:

Aus $7^2+9^2 = 3^2+11^2$, folgt bei $x=9-7$ die Identität

$$5, 7, 5, 13 \overset{n}{=} 1, 9, 9, 11 \qquad (n=1, 3);$$

aus $1^2+8^2=4^2+7^2$ folgt bei $x=8-1$ die Relation

$$1, 3, 11, 14 \overset{n}{=} 6, 8, 15, \qquad (n=1, 3);$$

aus $2^2+16^2=8^2+14^2$ folgt bei $x=\frac{2+16}{2}$ die Identität

$$1, 17, 23 \overset{n}{=} 5, 11, 25, \qquad (n=1, 3);$$

bei $7^2+6^2=2^2+9^2$ ergibt $x=2$ das Resultat

$$5, 4, 4, 11 \overset{n}{=} 7, 9, 8 \qquad (n=1, 3);$$

aus $7^2+9^2=3^2+11^2$ wird bei $x=\frac{7+9}{2}$ das Resultat

$$2,\ 2,\ 3,\ 6,\ 13 \overset{n}{=} 5,\ 10,\ 11 \qquad (n=1,\ 3).$$

(*c*) Interessantere Fälle ergeben sich, wenn man von der Gleichung

$$A_1^2+A_2^2 = B_1^2+B_2^2,$$

wobei $A_1=A_2$, ausgeht.

Setzt man hier $x=A_1=A_2$, so bekommt man Relationen von der Form

$$X_1,\ X_2,\ X_3 \overset{n}{=} Y_1,\ Y_2,\ Y_3 \qquad (n=1,\ 3).$$

So wird aus $5^2+5^2=1^2+7^2$ bei $x=5$ die Identität

$$2,\ 3,\ 6 \overset{n}{=} 1,\ 5,\ 5 \qquad (n=1,\ 3).$$

Geht man von der Gleichung

$$A_1^2+A_2^2 = B_1^2+B_2^2,$$

wobei $B_1=0$ ist, also von einer pythagoräischen Gleichung, aus und setzt $x=B_2$, so kommt man zu Relationen von der Form

$$U,\ U,\ 2U \overset{n}{=} Q,\ R,\ V,\ W \qquad (n=1,\ 3).$$

So folgt aus $3^2+4^2=0^2+5^2$ bei $x=5$ die Identität

$$5,\ 5,\ 10 \overset{n}{=} 1,\ 2,\ 8,\ 9 \qquad (n=1,\ 3).$$

Setzt man bei

$$A_1^2+A_2^2 = B_1^2+B_2^2,$$

wobei $B_1=0$ ist, $x=\frac{B_2}{2}$, so ergeben sich Relationen von der Form

$$M,\ N,\ P \overset{n}{=} S,\ T,\ Z \qquad (n=1,\ 3).$$

Eine Ausnahme macht $6^2+8^2=10^2$, denn hier ergibt $x=\frac{10}{2}$ die Relation

$$1,\ 3,\ 5,\ 15 \overset{n}{=} 11,\ 13 \qquad (n=1,\ 3).$$

Setzt man bei $A_1^2+A_2^2=B_1^2+B_2^2$ wobei $B_1=0$ ist, $x=\dfrac{A_1+A_2}{2}$,

bekommt man Relationen von der Form

$$G, G, H \overset{n}{=} J, K, L \qquad (n=1, 3).$$

Exempel: Aus $10^2+24^2=0^2+26^2$ ergibt $x=\dfrac{10+24}{2}$ die Relation

$$17, 17, 43 \overset{n}{=} 9, 27, 41 \qquad (n=1, 3).$$

(*d*) Wir setzten bei de Gleichung

$$A_1^2+A_2^2=B_1^2+B_2^2,$$

wobei $B_1=0$ ist, $x=\dfrac{A_1+B_2}{2}$ (Annahme: $A_2>A_1$) und erhalten

dann Identitäten von der Form

$$A^2+A^2+B+C = D^2+D^2+E+F,$$

$$A^6+A^6+B^3+C^3=D^6+D^6+E^3+F^3.$$

Exempel: $10^2+24^2=0^2+26^2$ ergibt $x=18$ und man bekommt

$$8^2+8^2+8+22 = 2^2+2^2+14+21,$$

$$8^6+8^6+8^3+22^3=2^6+2^6+14^3+21^3.$$

Eine singuläre Lösung ergibt $3^2+4^2=0^2+5^2$, nämlich

$$2^2+2^2+3^2=1^2+1^2+7+8,$$

$$2^6+2^6+3^6=1^6+1^6+7^3+8^3.$$

II. Wir wenden den Satz unter (I*a*) auf die Relation $G, G \overset{n}{=} J, K, L$, für $n=2$ und 4 an. (N.B. Hiebei muss stets $J+K=L$ sein.). Wir schreiben die Relation analog gewissen Fällen unter I in dieser Form:

$$0, G, G \overset{n}{=} J, K, L \text{ für } n=2, 4.$$

(*a*) Wir setzen $x=2G$ und bekommen

$$-2G,\ -G,\ -G,\ J+2G,\ K+2G,\ L+2G$$

$$\overset{n}{=}\ J+2G,\ K+2G,\ L+2G,\ 2G,\ 3G,\ 3G,\ \text{für } 1, 3, 5.$$

Es ergeben sich Identitäten von der Form

$$M_1,\ M_2,\ M_3,\ M_4,\ M_5,\ M_6 \overset{n}{=} 2\,(N,\ P,\ Q)$$

für $n=1, 2, 3, 4, 5$, wenn man die negativen Glieder auf die entgegengesetzte Seite schafft.

So bekommt man aus

$$0,\ 7,\ 7 \overset{n}{=} 3,\ 5,\ 8,$$

für $n=2, 4$, wenn $x=2.7$ ist, die Identität

$$6,\ 9,\ 11,\ 17,\ 19,\ 22 \overset{n}{=} 2\,(7,\ 14,\ 21) \text{ für } n=1,\ 2,\ 3,\ 4,\ 5.$$

N.B. Bei der Identität

$$M_1,\ M_2, \ldots\ldots, M_6 \overset{n}{=} 2\,(N,\ P,\ Q) \text{ für } n=1,\ 2,\ 3,\ 4,\ 5,$$

bestehen noch die Beziehungen $P=2N$ und $Q=3N$, ferner:

$$M_1+M_6 = M_2+M_5 = M_3+M_4=N+Q = 2P.$$

(*b*) Setzt man bei $0,\ G,\ G \overset{n}{=} J,\ K,\ L$ für $n=2, 4$ bei Anwendung des Satzes unter (I*a*) für $x=G$, so bekommt man Identitäten von der Form

$$a,\ b,\ c,\ d,\ e \overset{n}{=} f,\ G,\ G,\ 2G,\ 2G \qquad \text{für } n=1,\ 3,\ 5.$$

(*c*) Setzt man bei $0,\ G,\ G \overset{n}{=} J,\ K,\ L$, für $n=2, 4$ bei Anwendung des

erwähnten Satzes für $x=\dfrac{G-K}{2}$, so bekommen wir Identitäten von der Form

$$g,\ g,\ g,\ h,\ j \overset{n}{=} m,\ m,\ n,\ p,\ q,\ r,\ r \text{ für } n=1,\ 3,\ 5.$$

wenn man die negativen Glieder auf die entgegengesetzte Seite schafft. Da sich beiderseits ein Glied streichen lässt, wenn man bei

$$0, 7, 7 \overset{n}{=} 3, 5, 8 \qquad \text{für } n=2, 4$$

den Satz unter (I a) anwendet und $x=\frac{7-5}{2}$ setzt, so ergibt sich aus diesem numerischen Exempel die merkwürdige (singuläre ?) Identität:

$$6, 6, 6, 9 \overset{n}{=} 1, 1, 2, 7, 8, 8 \qquad \text{für } n=1, 3, 5.$$

(d) Wir setzen bei

$$0, 2G, 2G \overset{n}{=} 2J, 2K, 2L \quad \text{für } n=2, 4, \quad \text{für } x=2J+K,$$

und wenden unseren Satz an, dann bekommen wir, wenn wir die negativen Glieder auf die entgegengesetzte Seite schaffen, Identitäten von der Form

$$q, r, r, s, t, u \overset{n}{=} 2(v, w) \text{ für } n=1, 3, 5.$$

Beispiel: Bei

$$0, 26, 26 \overset{n}{=} 14, 16, 30 \text{ für } n=2, 4$$

ergibt sich $x=14+8=22$ und wir bekommen das numerische Exempel

$$3, 2, 2, 18, 19, 26 \overset{n}{=} 2(11, 24) \qquad \text{für } n=1, 3, 5,$$

nachdem wir den gemeinsamen Faktor 2^n gestrichen haben.

Bemerkung: Wir haben durch Anwendung eines *einzigen* Satzes in elementarer Methode ähnliche Identitäten entwickelt, wie sie A. Golden* fand.

Nurnberg, Germany.

* *Cf.* Nieuw Archief voor Wiskunde, 1935, 85, (Edition Noordhoff, Groningen i. Holland).

Rings with Non-Commutative Addition

By

Olga Taussky.

In a paper to appear in Monatshefte für Mathematik und Physik Ph. Furtwängler and I proved that a ring which contains an element which is not a zero divisor has necessarily commutative addition.

The term ring is used for an (not necessarily abelian) additively written group G for whose elements a second composition (called multiplication) is defined such that

$$a(b+c) = ab+ac,$$
$$(b+c)a = ba+ca.$$

We shall prove a theorem which generalises the fact mentioned above:

Theorem I. Any two elements of a ring which are products are permutable.

Proof. Let us denote the additive group of the ring by G. Let a, b, c, d be arbitrary elements of G. According to the distributive laws we can compute the product

$$(a+b)(c+d)$$

in two different ways. We obtain thus

$$ac+bc+ad+bd = ac+ad+bc+bd,$$

hence $$bc+ad = ad+bc.$$

If we put $$a = b,$$

we obtain $$a(c+d) = a(d+c),$$

or $$a(c+d-c-d) = 0.$$

From this, the theorem mentioned at the beginning follows. For, if there exists a non-zero divisor, a in G, $c+d-c-d$ is necessarily zero.

We can even say: *If there exists an element which is not both left and right zero-divisor, G is necessarily abelian.* For, If a is left zero-divisor, but not right zero-divisor, we consider the expression

$$(c+d)(a+b)$$

and put $$a = b.$$

It follows $$(c+d-c-d)a = 0,$$

hence $$c+d = d+c.$$

Hence: *If G is not abelian, every commutator and therefore every element c of the commutator sub-group G' is absolute zero divisor, i.e., $cg = gc = 0$ for every $g \in G$.**

From the distributive laws follows then: a product is not altered if one adds an element of G' to one of its factors. From this follows

Theorem II. The multiplication of G induces a multiplication for the residue-classes with respect to G'. $G-G'$ is, of course, a ring with commutative addition.

As the elements of G which are products are permutable among themselves, they generate an abelian sub-group P of G. Let us call I the group which is the intersection of P and G'. If $I=0$, no element of G' except 0 is a product and every residue-class with respect to G' contains at most one element which is a product. In what follows we shall consider rings such that $I=0$.

Theorem III. Let $G-G'$ be a ring. The multiplication of $G-G'$ can be extended to multiplication for G such that $I=0$, if and only if in every class of $G-G'$, which is a product, a representative can be chosen such that these elements generate a group which has no intersection with G'.

Proof. The condition mentioned in the theorem is obviously necessary.

We prove now that the condition is sufficient. Let there exist a set of elements representing the residue-classes with respect to G', such that the group generated by the elements which represent products has no intersection with G'. Let us define the product of any two elements a, b of G as the element which represents the product of the classes defined by a and b. We have only to prove that the distributive laws hold for this multiplication. Let a, b, c be any elements of G. Since the distributive law holds for $G-G'$, the elements $a(b+c)$ and $ab+ac$ are in the same class with respect to G'. Hence, they can only differ by an element of G'. But $a(b+c)-ab-ac$ is an element of the group generated by products. Hence it is 0. In the same way the other distributive law follows.

* If G coincides with G' it can only be a ring, where $ab=0$ for every $a, b \in G$. The author is indebted to Mr. H. Mann for a helpful remark.

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